Graph Database

Jason Grout

December 28, 2005

Here are some problems that are particularly suited to exploring on the graph database (

`http://math.byu.edu/~grout/graphs/`

)

1 General Graph Information

1. How many edges does a complete graph on n vertices have?
2. How many edges does a tree on n vertices have?
3. How many edges does a forest with n vertices and p components have?
4. Given two drawings of graphs with 5 vertices and 9 edges, are they necessarily isomorphic? How about 5 vertices and 8 edges? If they aren't isomorphic, then what is one distinguishing characteristic about each of them?
5. What's the largest clique in a given graph? What's the largest independent set in a given graph?
6. What is the girth or radius of some given graphs?
7. Conjecture a pattern in the spectrum of bipartite graphs.

2 Degrees

1. Conjecture and prove a theorem about the number of vertices of odd degree. (There is an even number of odd-degree vertices) [Hartsfield and Ringel, 1990, p. 12].
2. Conjecture and prove a theorem about the multiplicity of degrees. (There is always at least one degree that is repeated) [Hartsfield and Ringel, 1990, p. 12].
3. Conjecture and prove theorems about the number of cycles in a connected graph when:
   (a) the average degree of G is less than 2.
   (b) the average degree of G is equal to 2.

[Hartsfield and Ringel, 1990, p. 20].

4. Let G be a tree and suppose that the degrees of vertices in G are odd. Conjecture and prove a theorem about the number of edges. (The number of edges is odd) [Hartsfield and Ringel, 1990, p. 21].
5. Suppose G is a tree with $n + 1$ vertices and exactly two vertices of degree 1. Conjecture and prove a theorem about the structure of G. (G is a path) [Hartsfield and Ringel, 1990, p. 21].
6. Find a relationship between the sum of the degrees and the number of edges in a graph.
7. Find a relationship between the degrees and whether a graph is Eulerian.
8. Find a relationship between the degrees of the vertices and whether the graph is Hamiltonian. Assume that the number of vertices in G is greater than 2. (If each degree is at least $n/2$, then G is Hamiltonian). [Rosen, 1994, p. 481].
9. Conjecture and prove a statement about the multiplicity of degrees in a graph.
10. Does the degree sequence determine the graph?
11. What's the maximum degree and minimum degree of a certain given graph?
12. How many edges does a regular graph have?
13. Is a given graph regular?
14. Characterize the two-regular graphs (i.e., all vertices have degree 2).

3 Connectivity

Let G be a connected graph.

1. Let p be the number of vertices in G and q be the number of edges in G. Conjecture and prove an inequality between p and q. ($p \leq q + 1$) [Hartsfield and Ringel, 1990, p.17]
2. Suppose G has more vertices than edges. Conjecture and prove a theorem about the structure of G. (G is a tree) [Hartsfield and Ringel, 1990, p. 21].
3. Can a regular, connected graph have a cut vertex? (First example has 10 vertices!)
4. Find a relationship between the number of edges and the connectivity of the graph.
5. What is the vertex and edge connectivity of a tree?
6. What is the edge connectivity of a clique?

4 Complements

1. Prove a relationship between the number of edges in a graph and the number of edges in the graph's complement.
2. Find a relationship between the clique number of a graph G and the independence number of the complement of G.

5 Chromatic Number

1. Determine the chromatic number of a cycle. [Hartsfield and Ringel, 1990, p. 29].
2. Prove or disprove an inequality between the chromatic number and the independence number. (Make a table of both numbers).
3. Prove or disprove an inequality involving the clique number and the chromatic number?
4. Conjecture a relationship between the chromatic number and whether a graph is bipartite.
5. What is the chromatic number of a tree?
6. Let k be a positive integer. Does there exist a graph G so that the clique number of G is 2 and the chromatic number is k? [Roberts, 1984, p. 111].
7. Conjecture and prove an inequality between the maximum degree and the chromatic number of a graph. [Roberts, 1984, p. 110].
8. Conjecture and prove both an upper and lower bound for the chromatic number of G which involves the independence number $\alpha(G)$. ($\frac{n}{\alpha(G)} \leq \chi(G) \leq n - \alpha(G) + 1$.) [Roberts, 1984, p. 110].

6 Planarity

1. Suppose G has at most 3 cycles. Is G planar? [Hartsfield and Ringel, 1990, p. 151].
2. Conjecture (and attempt to prove :) an upper bound on the chromatic number of planar graphs. [Hartsfield and Ringel, 1990, pp. 160–161].
3. Let G be a connected planar graph on 3 or more vertices. Prove or disprove an inequality between the number of edges and the number of vertices. (Hint: Use Euler's formula). ($e \leq 3v - 6$) [Rosen, 1994, p. 503].
4. Let G be a connected planar graph on 3 or more vertices. Suppose G has no cycles of length 3 (i.e., G is triangle-free). Prove or disprove an inequality between the number of edges and the number of vertices. (Hint: Use Euler's formula). ($e \leq 2v - 4$) [Rosen, 1994, p. 504].

7 Cycles

1. Find a relationship between the number of cycles and the number of edges.
2. How many spanning trees are in the cycle C_n? [Hartsfield and Ringel, 1990, p. 99].
3. Find the number of Hamiltonian cycles in a wheel. [Hartsfield and Ringel, 1990, p. 42].

8 Induced and Forbidden Subgraphs

1. Find a list of subgraphs that imply that a graph is nonplanar.
2. Is this particular graph chordal?
3. Characterize the graphs with at most three edges and no isolated vertices. How many are there? Can you find a list of forbidden subgraphs characterizing this class?
4. Characterize the graphs that have P_3 as a forbidden subgraph.
5. Characterize the cycle-free graphs.

References

Nora Hartsfield and Gerhard Ringel. *Pearls in graph theory.* Academic Press Inc., Boston, MA, 1990. ISBN 0-12-328552-6. A comprehensive introduction.

Fred S. Roberts. *Applied combinatorics.* Prentice Hall Inc., Englewood Cliffs, NJ, 1984. ISBN 0-13-039313-4.

Rosen. *Discrete Mathematics and its Applications.* Third Edition, 1994.