The minimum rank problem over the finite field of order 2: minimum rank 3

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Abstract

Our main result is a sharp bound for the number of vertices in a minimal forbidden subgraph for the graphs having minimum rank at most 3 over the finite field of order 2. We also list all 62 such minimal forbidden subgraphs. We conclude by exploring how some of these results over the finite field of order 2 extend to arbitrary fields and demonstrate that at least one third of the 62 are minimal forbidden subgraphs over an arbitrary field for the class of graphs having minimum rank at most 3 in that field.

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1 Introduction

Given a field F and a simple undirected graph G on n vertices, let $S(F, G)$ be the set of symmetric $n \times n$ matrices A with entries in F satisfying $a_{ij} \neq 0$, $i \neq j$, if and only if ij is an edge in G. There

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is no restriction on the diagonal entries of the matrices in $S(F,G)$. Let

$$\mathrm{mr}(F,G) = \min\{\mathrm{rank}\ A \mid A \in S(F,G)\}.$$

The problem of finding $\mathrm{mr}(F,G)$ has recently attracted considerable attention, particularly for the case in which $F = \mathbb{R}$ (see the survey paper [FH07] and the references cited there or the American Institute of Mathematics workshop website [Ame06] on this topic). Relevant papers for us are [Hsi01, BFH04, BvdHL04, BD05, BvdHL05, DK06]. In [BvdHL04] and [BvdHL05], the problem of characterizing all graphs G for which $\mathrm{mr}(F,G) \leq 2$ was addressed. Complete characterizations were obtained for all fields and fall into four cases depending on whether the field is infinite or finite and whether or not the field characteristic is two. These various classifications have both striking similarities and distinctive differences.

The full house, seen and labeled in Figure 1, is the only graph on 5 or fewer vertices for which the field affects the minimum rank. (This was previously noted in [BvdHL05], in which the graph was identified as $\overline{P_3 \cup 2K_1}$.)

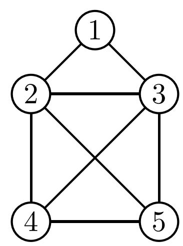


Figure 1: The full house graph.

Proposition 1 ([BvdHL05])**.** *Let G be a graph on 5 or fewer vertices and suppose that $G \neq$ full house. Then $\mathrm{mr}(F,G)$ is independent of the field F.*

We also include a short proof for the fact that the minimum rank of the full house graph is field-dependent.

Proposition 2. *If $F \neq \mathbb{F}_2$ is a field, then* $\mathrm{mr}(F, \text{full house}) = 2$*. However,* $\mathrm{mr}(\mathbb{F}_2, \text{full house}) = 3$*.*

Proof. If $F \neq \mathbb{F}_2$, there are elements $a, b \neq 0$ in F such that $a + b \neq 0$. Then

$$\begin{bmatrix} a & a & a & 0 & 0 \\ a & a+b & a+b & b & b \\ a & a+b & a+b & b & b \\ 0 & b & b & b & b \\ 0 & b & b & b & b \end{bmatrix} \in S(F, \text{full house}),$$

which shows that $\mathrm{mr}(F, \text{full house}) = 2$. But if

$$A = \begin{bmatrix} d_1 & 1 & 1 & 0 & 0 \\ 1 & d_2 & 1 & 1 & 1 \\ 1 & 1 & d_3 & 1 & 1 \\ 0 & 1 & 1 & d_4 & 1 \\ 0 & 1 & 1 & 1 & d_5 \end{bmatrix}$$

is any matrix in $S(\mathbb{F}_2, \text{full house})$, then

$$\det A[\{1,2,5\},\{1,3,4\}] = \begin{vmatrix} d_1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 1,$$

so $\operatorname{mr}(\mathbb{F}_2, \text{full house}) \geq 3$. Setting all $d_i = 1$ verifies that $\operatorname{mr}(\mathbb{F}_2, \text{full house}) = 3$. □

In spite of this dependence on the field, it has become clear that even for calculating the minimum rank over just the real field, results obtained over finite fields, and over $\mathbb{F}_2$ in particular, will provide important insights. This will be explored more fully in Section 11.

The methods of [BvdHL04, BvdHL05] do not extend in any straightforward way to the problem of characterizing graphs with $\operatorname{mr}(F, G) \leq k$ for $k \geq 3$. However, it is possible to obtain results of this sort for finite fields using other methods which make explicit use of the finiteness of F. In this paper we examine the case $F = \mathbb{F}_2$.

We recall some notation from graph theory.

Definition 3. Given a graph G, $V(G)$ denotes the set of vertices in G and $E(G)$ denotes the set of edges in G. The *order* of a graph is $|G| = |V(G)|$. Given two graphs G and H, with $V(G)$ and $V(H)$ disjoint, the *union* of G and H is $G \cup H = (V(G) \cup V(H), E(G) \cup E(H))$. The *join*, $G \vee H$, is the graph obtained from $G \cup H$ by adding edges from all vertices of G to all vertices of H. If $S \subset V(G)$, $G[S]$ denotes the subgraph of G induced by S. If H is an induced subgraph of G, $G - H$ denotes the subgraph induced by $V(G) \setminus V(H)$.

Definition 4. We denote the path on n vertices by P_n. The complete graph on n vertices will be denoted by K_n. We abbreviate $K_n \cup \cdots \cup K_n$ (m times) to mK_n.

We recall the following observation.

Observation 5 ([BvdHL04, Observation 5])**.** If H is an induced subgraph of G, then for any field F, $\operatorname{mr}(F, H) \leq \operatorname{mr}(F, G)$.

Example 6. It is well known that $\operatorname{mr}(F, P_{k+2}) = k + 1$ for any field F. Therefore P_{k+2} cannot be an induced subgraph of any graph G with $\operatorname{mr}(F, G) \leq k$.

Definition 7. Let F be any field. The graph H is a *minimal forbidden subgraph* for the class of graphs $\mathcal{G}_k(F) = \{G \mid \operatorname{mr}(F, G) \leq k\}$ if

(a) $\operatorname{mr}(F, H) \geq k + 1$ and

(b) $\operatorname{mr}(F, H - v) \leq k$ for every vertex $v \in V(H)$.

Let $\mathcal{F}_{k+1}(F)$ be the set of all minimal forbidden subgraphs for $\mathcal{G}_k(F)$.

Observation 8. $G \in \mathcal{G}_k(F) \iff$ no graph in $\mathcal{F}_{k+1}(F)$ is induced in G.

Theorem 6 ($a \iff c$) of [BvdHL04] and Theorem 16 of [BvdHL05] can be restated:

Theorem 9 ([BvdHL04, Theorem 6])**.** $\mathcal{F}_3(\mathbb{R}) = \{P_4, \ltimes, \text{dart}, P_3 \cup K_2, 3K_2, K_{3,3,3}\}$.

Theorem 10 ([BvdHL05, Theorem 16])**.**

$$\mathcal{F}_3(\mathbb{F}_2) = \{P_4, \ltimes, \text{dart}, P_3 \cup K_2, 3K_2, \text{full house}, P_3 \vee P_3\}.$$

Ding and Kotlov [DK06] obtained an important result related to $\mathcal{F}_{k+1}(F)$. They showed that if F is a finite field, then every graph in $\mathcal{F}_{k+1}(F)$ has at most $\left(\frac{|F|^k}{2}+1\right)^2$ vertices, so, in particular, $\mathcal{F}_{k+1}(F)$ is finite. In the special case $k=3$ and $F=\mathbb{F}_2$, their result implies that each graph in $\mathcal{F}_4(\mathbb{F}_2)$ has at most 25 vertices. In this paper, we improve their bound for this case to show that every graph in $\mathcal{F}_4(\mathbb{F}_2)$ has at most 8 vertices. This new bound makes an exhaustive computer search feasible, which gives the result that $|\mathcal{F}_4(\mathbb{F}_2)| = 62$ and also shows that the new bound is sharp. Of the 29 graphs in $\mathcal{F}_4(\mathbb{F}_2)$ having vertex connectivity at most one, we will prove that 21 graphs are in $\mathcal{F}_4(F)$ for every field F, while none of the remaining 8 graphs are in $\mathcal{F}_4(F)$ for any field $F \neq \mathbb{F}_2$.

Our approach relies on the following generalization of $\mathcal{F}_{k+1}(F)$.

Definition 11. Given a field F and a graph H, let $\mathcal{F}_{k+1}(F,H)$ be the set of graphs G containing H as an induced subgraph and satisfying

(a) $\operatorname{mr}(F,G) \geq k+1$ and

(b) for some H induced in G, $\operatorname{mr}(F,G-v) \leq k$ for every $v \in V(G-H)$.

Example 12. Let F be any field, let G be the graph labeled in Figure 2, let $H = P_4$, and let

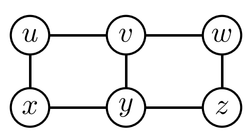
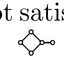


Figure 2: G in Example 12.

$k=3$. Since P_5 is induced in G, $\operatorname{mr}(F,G) \geq 3+1$, so condition (a) is satisfied.

Six copies of $H=P_4$ are induced in G. For $H = G[\{u,v,w,x\}]$, we have $\operatorname{mr}(F,G-y)=4$, so condition (b) is not satisfied for this copy of P_4. However, if $H = G[\{u,v,y,z\}]$, both $G-w$ and $G-x$ are isomorphic to , which has minimum rank 3 by Theorem 2.3 in [BFH04] (see Theorem 57 in this paper). Therefore condition (b) is satisfied for this induced P_4, so $G \in \mathcal{F}_{k+1}(F,P_4)$.

In the notation of Definition 11, $\mathcal{F}_{k+1}(F) = \mathcal{F}_{k+1}(F,\emptyset)$, where $\emptyset$ is the empty graph.

Theorem 13.

$$\mathcal{F}_{k+1}(F) \subseteq \bigcup_{H \in \mathcal{F}_k(F)} \mathcal{F}_{k+1}(F,H).$$

Proof. Let $G \in \mathcal{F}_{k+1}(F)$. Since $\operatorname{mr}(F,G) \geq k+1 > k-1$, $G \notin \mathcal{G}_{k-1}(F)$. Therefore some graph $H \in \mathcal{F}_k(F)$ is induced in G. By definition, $\operatorname{mr}(F,G-v) \leq k$ for every vertex v of G, so $\operatorname{mr}(F,G-v) \leq k$ for every vertex v of $G-H$. By definition, $G \in \mathcal{F}_{k+1}(F,H)$. $\square$

Combining Theorems 10 and 13, we have the following result.

Corollary 14.

$$\begin{aligned}\mathcal{F}_4(\mathbb{F}_2) \subseteq &\bigcup_{H\in\mathcal{F}_3(\mathbb{F}_2)} \mathcal{F}_4(\mathbb{F}_2, H)\\ &= \mathcal{F}_4(\mathbb{F}_2, 3K_2) \cup \mathcal{F}_4(\mathbb{F}_2, P_3 \vee P_3) \cup \mathcal{F}_4(\mathbb{F}_2, \text{dart}) \cup \mathcal{F}_4(\mathbb{F}_2, \ltimes)\\ &\quad \cup \mathcal{F}_4(\mathbb{F}_2, P_3 \cup K_2) \cup \mathcal{F}_4(\mathbb{F}_2, \text{full house}) \cup \mathcal{F}_4(\mathbb{F}_2, P_4).\end{aligned}$$

Sections 2–10 are devoted to explicitly determining $\mathcal{F}_4(\mathbb{F}_2)$.

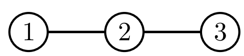
2 Matrices which attain a minimum rank for $\mathcal{F}_3(\mathbb{F}_2)$

Given a field F and a graph G, it is natural to seek to determine all matrices in $S(F, G)$ which attain the minimum rank of G. Determining these matrices plays a critical role in determining $\mathcal{F}_4(\mathbb{F}_2)$.

Definition 15. Let G be a graph. Let $\mathcal{MR}(F, G) = \{A \in S(F, G) \mid \text{rank } A = \text{mr}(F, G)\}$, the set of matrices in $S(F, G)$ that attain the minimum rank of G. Call two matrices in $\mathcal{MR}(F, G)$ *equivalent* if and only if they have the same column space. Let $\mathcal{C}(F, G)$ be the resulting set of equivalence classes.

Let G be a graph. In the remainder of this section and in Sections 3–9, we will assume that $F = \mathbb{F}_2$ and abbreviate our notation as follows: $S(\mathbb{F}_2, G)$ is shortened to $S(G)$, $\text{mr}(\mathbb{F}_2, G)$ is shortened to $\text{mr}(G)$, $\mathcal{F}_{k+1}(\mathbb{F}_2)$ is shortened to $\mathcal{F}_{k+1}$, $\mathcal{F}_{k+1}(\mathbb{F}_2, G)$ is shortened to $\mathcal{F}_{k+1}(G)$, $\mathcal{MR}(\mathbb{F}_2, G)$ is shortened to $\mathcal{MR}(G)$, and $\mathcal{C}(\mathbb{F}_2, G)$ is shortened to $\mathcal{C}(G)$.

In the remainder of this section, we determine $\mathcal{MR}(G)$ for all of the graphs in $\mathcal{F}_3$ (see Theorem 10).

Lemma 16. *With P_3 labeled as* 

,

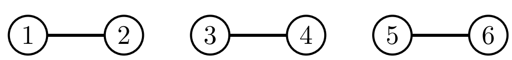
$$\mathcal{MR}(P_3) = \left\{ \begin{bmatrix}0&1&0\\1&0&1\\0&1&0\end{bmatrix}, \begin{bmatrix}0&1&0\\1&1&1\\0&1&0\end{bmatrix}, \begin{bmatrix}1&1&0\\1&0&1\\0&1&1\end{bmatrix} \right\}.$$

Proof. Since $\text{mr}(P_3) = 2$,

$$A = \begin{bmatrix}x&1&0\\1&y&1\\0&1&z\end{bmatrix} \in \mathcal{MR}(P_3) \iff \det A = xyz + x + z = 0 \ \text{ in } \mathbb{F}_2.$$

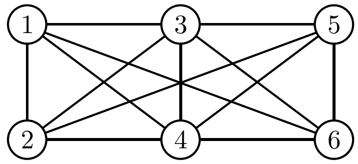
If $x \neq z$, then $\det A = 1$, so $x = z$. Then $\det A = xy$, so $A \in \mathcal{MR}(P_3)$ if and only if either $x = y = z = 0$, $x = z = 0$ and $y = 1$, or $x = z = 1$ and $y = 0$. □

Proposition 17. *The sets $\mathcal{MR}(G)$ for $G \in \mathcal{F}_3$ are as follows.*

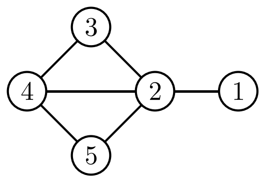
(a) With $3K_2$ labeled as 

,

$$\mathcal{MR}(3K_2) = \left\{ \begin{bmatrix}1&1\\1&1\end{bmatrix} \oplus \begin{bmatrix}1&1\\1&1\end{bmatrix} \oplus \begin{bmatrix}1&1\\1&1\end{bmatrix} \right\}.$$

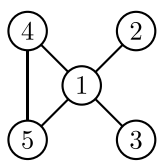
(b) *With $P_3 \vee P_3$ labeled as*  ,

$$\mathcal{MR}(P_3 \vee P_3) = \left\{ \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 \end{bmatrix} \right\}.$$

(c) *With the dart labeled as*  ,

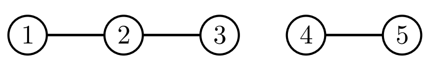
$$\mathcal{MR}(\text{dart}) = \left\{ M_1 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}, M_2 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} \right\}$$

and $\mathcal{C}(\text{dart}) = \{C_1 = \{M_1\}, C_2 = \{M_2\}\}$.

(d) *With $\ltimes$ labeled as*  ,

$$\mathcal{MR}(\ltimes) = \left\{ M_1 = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \end{bmatrix}, M_2 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \end{bmatrix}, M_3 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \end{bmatrix} \right\}$$

and $\mathcal{C}(\ltimes) = \{C_1 = \{M_1, M_2\}, C_2 = \{M_3\}\}$.

(e) *With $P_3 \cup K_2$ labeled as*  ,

$$\mathcal{MR}(P_3 \cup K_2) = \left\{ M_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, M_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \right.$$
$$\left. M_3 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \oplus \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$$

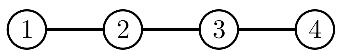
and $\mathcal{C}(P_3 \cup K_2) = \{C_1 = \{M_1, M_2\},\, C_2 = \{M_3\}\}$.

(f) *With the full house labeled as in Figure 1,*

$$\mathcal{MR}(\text{full house}) = \left\{ M_1 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix},\ M_2 = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}, \right.$$

$$\left. M_3 = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix},\ M_4 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} \right\}$$

and $\mathcal{C}(\text{full house}) = \{C_1 = \{M_1, M_2\},\, C_2 = \{M_3\},\, C_3 = \{M_4\}\}$.

(g) *With* P_4 *labeled as*  ,

$$\mathcal{MR}(P_4) = \left\{ M_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix},\ M_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \right.$$

$$\left. M_3 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix},\ M_4 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix},\ M_5 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \right\}$$

and $\mathcal{C}(P_4) = \{C_1 = \{M_1, M_2\},\, C_2 = \{M_3, M_4\},\, C_3 = \{M_5\}\}$.

Proof 1. Exhaustively calculate the rank of each matrix in $S(G)$ for each $G \in \mathcal{F}_3$. Appendix A contains a collection of Magma [BCP97] functions to implement this approach. □

Proof 2. It is known [BvdHL04, BvdHL05] that each of the graphs in (a) through (g) has minimum rank 3.

Part (a) follows immediately and (e) follows from Lemma 16. We prove (f) and (g). The proofs of (b), (c), and (d) are similar.

(f) Let

$$A = \begin{bmatrix} v & 1 & 1 & 0 & 0 \\ 1 & w & 1 & 1 & 1 \\ 1 & 1 & x & 1 & 1 \\ 0 & 1 & 1 & y & 1 \\ 0 & 1 & 1 & 1 & z \end{bmatrix} \in \mathcal{MR}(\text{full house}).$$

I. $v = 0$. Elementary row and column operations reduce A to

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & w & 1 & 0 & 0 \\ 1 & 1 & x & 0 & 0 \\ 0 & 0 & 0 & y & 1 \\ 0 & 0 & 0 & 1 & z \end{bmatrix}.$$

Then we must have $y = z = 1$. Since $\begin{vmatrix} 0 & 1 & 1 \\ 1 & w & 1 \\ 1 & 1 & x \end{vmatrix} = w + x$, we must have $w = x$, so $w = x = 1$ or $w = x = 0$. This yields the matrices M_2 and M_3 in (f).

II. $v = 1$. Row and column reductions yield

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & w+1 & 0 & 1 & 0 \\ 0 & 0 & x+1 & 1 & 0 \\ 0 & 1 & 1 & y & y+1 \\ 0 & 0 & 0 & y+1 & y+z \end{bmatrix}.$$

If $w = 0$, then B can be further reduced to

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & x+1 & 1 & 0 \\ 0 & 0 & 1 & y+1 & y+1 \\ 0 & 0 & 0 & y+1 & y+z \end{bmatrix},$$

which has rank at least 4, so we must have $w = 1$. Then

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & x+1 & 1 & 0 \\ 0 & 1 & 1 & y & y+1 \\ 0 & 0 & 0 & y+1 & y+z \end{bmatrix},$$

which reduces to

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & x+1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & y+z \end{bmatrix}.$$

In order for rank $C = 3$, we require that $x = 1$ and $y = z$. This yields matrices M_1 and M_4 in (f).

(g) Let

$$A = \begin{bmatrix} w & 1 & 0 & 0 \\ 1 & x & 1 & 0 \\ 0 & 1 & y & 1 \\ 0 & 0 & 1 & z \end{bmatrix} \in \mathcal{MR}(P_4).$$

If $w = 0$, by elementary row and column operations the matrix reduces to

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & y & 1 \\ 0 & 0 & 1 & z \end{bmatrix}.$$

In order for rank $B = 3$, we must have $y = z = 1$, but x can be 0 or 1. This yields matrices M_1 and M_2 in (g). If $w = 1$, one row and column operation gives

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & x+1 & 1 & 0 \\ 0 & 1 & y & 1 \\ 0 & 0 & 1 & z \end{bmatrix}.$$

In order for C to have rank 3,

$$\begin{bmatrix} x+1 & 1 & 0 \\ 1 & y & 1 \\ 0 & 1 & z \end{bmatrix} \text{ must be in } \mathcal{MR}(P_3),$$

which by Lemma 16 gives three possibilities for x, y, and z, giving matrices M_3, M_4, and M_5 in (g).

□

3 General theorems

Throughout this section, let G be a graph with an induced subgraph H such that $\text{mr}(H) = k$.

For convenience, in sections 3–9, we will consider G as a complete graph with weighted edges. The weight of an edge, $\text{wt}(ij)$, is 1 if ij is an edge in the original graph and 0 if it is not. The vertices in $G - H$ will also have weights. Let the vertices of H be labeled $h_1, h_2, \ldots, h_\ell$. The weight $\text{wt}(v)$ of a vertex $v \in V(G - H)$ is the vector $(\text{wt}(vh_1), \text{wt}(vh_2), \ldots, \text{wt}(vh_\ell))^T$ of edge weights between the vertex v and the vertices of H.

3.1 Definitions

Definition 18. Let $M \in \mathcal{MR}(H)$. We say the vertex v in $G - H$ is *rank-preserving* with respect to M if

$$\text{rank} \begin{bmatrix} M & \text{wt}(v) \end{bmatrix} = \text{rank } M.$$

If v is rank-preserving with respect to M, then M can be augmented by a row and column to obtain a matrix in $S(G[V(H) \cup \{v\}])$ of rank k, so $\text{mr}(G[V(H) \cup \{v\}]) = \text{mr}(H)$. If v is not rank-preserving with respect to M, we say v is *rank-increasing* with respect to M. We say that a set of vertices is *rank-preserving* with respect to M if each vertex is rank-preserving with respect to M, and a set is *rank-increasing* with respect to M if some vertex is rank-increasing with respect to M.

Definition 19. Let $M \in \mathcal{MR}(H)$. We say the edge $uv \in G - H$, $u \neq v$, is *rank-preserving* with respect to M if u and v are rank-preserving with respect to M and $\mathrm{wt}(uv)$ is the unique number that satisfies the equality

$$\operatorname{rank} \begin{bmatrix} M & \mathrm{wt}(u) \\ \mathrm{wt}(v)^T & \mathrm{wt}(uv) \end{bmatrix} = \operatorname{rank} M.$$

(If $\mathrm{wt}(u) = Mp$ and $\mathrm{wt}(v) = Mq$, then uv is rank-preserving if and only if $\mathrm{wt}(uv) = q^T Mp$.) If uv is not rank-preserving with respect to M, we say uv is *rank-increasing* with respect to M. Notice that uv is rank-preserving with respect to M if and only if $\mathrm{mr}(G[V(H) \cup \{u, v\}]) = \mathrm{mr}(H)$. We say that a set of edges is *rank-preserving* with respect to M if each edge is rank-preserving with respect to M and is *rank-increasing* with respect to M if some edge is rank-increasing with respect to M.

We emphasize one part of this definition as:

Observation 20. If a vertex $v \in G - H$ is rank-increasing with respect to M, then each edge incident to v in $G - H$ is also rank-increasing with respect to M.

Definition 21. Let $M \in \mathcal{MR}(H)$. Given an ordered set of vertex weights $v_1, \ldots, v_p \in \mathrm{col}(M)$, let $v_i = Ma_i$ and let $A = [a_1 \cdots a_p]$. Then we say that the $p \times p$ matrix $P = A^T MA$ is the *rank-preserving table* for the ordered set $v_1, \ldots, v_p$ with respect to M. Note that the ij entry of P is the edge weight needed to make the edge between two vertices with weights v_i and v_j a rank-preserving edge with respect to M.

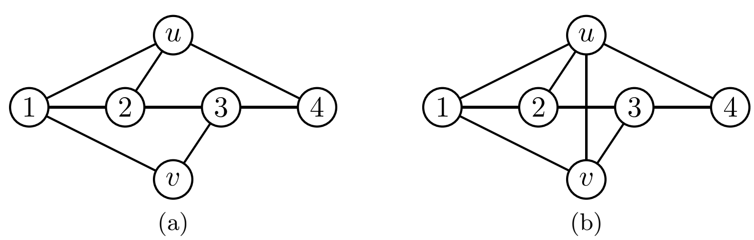


Figure 3: Graphs in Example 22.

Example 22. Let $H = P_4$, labeled as in Proposition 17, with corresponding $\mathcal{MR}(P_4)$ and $\mathcal{C}(P_4)$. Let G be a graph containing vertices $\{1, 2, 3, 4, u, v\}$ such that $H = G[\{1, 2, 3, 4\}]$ and $G[\{1, 2, 3, 4, u, v\}]$ is one of the graphs in Figure 3. Then u and v have weights $\mathrm{wt}(u) = (1, 1, 0, 1)^T$ and $\mathrm{wt}(v) = (1, 0, 1, 0)^T$. The vertex u is rank-preserving with respect to M_1 and M_2 since $\mathrm{wt}(u) \in \mathrm{col}(M_1) = \mathrm{col}(M_2)$ and is rank-increasing with respect to M_3, M_4, and M_5 since $\mathrm{wt}(u) \notin \mathrm{col}(M_3) = \mathrm{col}(M_4)$ and $\mathrm{wt}(u) \notin \mathrm{col}(M_5)$. Also, v is rank-preserving with respect to M_1, M_2, and M_5 and is rank-increasing with respect to M_3 and M_4. The set of vertices $\{u, v\}$ is rank-preserving with respect to M_1 and M_2 and is rank-increasing with respect to M_3, M_4, and M_5.

The edge uv is rank-increasing with respect to M_3, M_4, and M_5 because the set $\{u, v\}$ is rank-increasing with respect to each of those matrices. If $G[\{1, 2, 3, 4, u, v\}]$ is the graph in Figure 3(a), then $\mathrm{wt}(uv) = 0$ and uv is rank-preserving with respect to M_2 and rank-increasing with respect to M_1. If $G[\{1, 2, 3, 4, u, v\}]$ is the graph in Figure 3(b), then $\mathrm{wt}(uv) = 1$ and uv is rank-preserving with respect to M_1 and rank-increasing with respect to M_2. Rank-preserving tables with respect to M_1 and M_2 for $[\mathrm{wt}(u), \mathrm{wt}(v)]$ are, respectively,

$$P_1 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad P_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Note that $P_1 + P_2 = J$, the all-ones matrix. This property will be important later, so we give it a name now.

Definition 23. Two matrices A and B with entries in $\mathbb{F}_2$ are *complementary* if $A + B = J$, the all-ones matrix.

Definition 24. Let v be a vertex in $G - H$ and V be a set of vertices in $G - H$. Let

$$\mathcal{I}_v = \{M \in \mathcal{MR}(H) \mid v \text{ is rank-increasing with respect to } M\}$$

and $\mathcal{I}_V = \cup_{v \in V} \mathcal{I}_v$, the set of matrices for which V is rank-increasing. Let

$$\bar{\mathcal{I}}_V = \{C \in \mathcal{C} \mid V \text{ is rank-increasing with respect to every } M \in C\}.$$

Let uv be an edge in $G - H$ and E be a set of edges in $G - H$. Let

$$\mathcal{I}_{uv} = \{M \in \mathcal{MR}(H) \mid uv \text{ is rank-increasing with respect to } M\}$$

and $\mathcal{I}_E = \cup_{uv \in E} \mathcal{I}_{uv}$, the set of matrices for which E is rank-increasing.

Example 25. We will continue from Example 22. We have $\mathcal{I}_u = \{M_3, M_4, M_5\}$ and $\bar{\mathcal{I}}_u = \{C_2, C_3\}$. We also have $\mathcal{I}_v = \{M_3, M_4\}$ and $\bar{\mathcal{I}}_v = \{C_2\}$.

If $\mathrm{wt}(uv) = 0$, as is pictured in Figure 3(a), then $\mathcal{I}_{uv} = \{M_1, M_3, M_4, M_5\}$. If $\mathrm{wt}(uv) = 1$, as is pictured in Figure 3(b), then $\mathcal{I}_{uv} = \{M_2, M_3, M_4, M_5\}$.

Observation 26. Let V' be a set of vertices in $G - H$ such that $\mathcal{I}_{V'} \neq \mathcal{MR}(H)$. Then for every $v \in V'$,

$$\mathrm{wt}(v) \in \bigcap_{M_i \in \mathcal{MR}(H) \setminus \mathcal{I}_{V'}} \mathrm{col}(M_i).$$

3.2 Theorems

Observation 27. We have $\mathrm{mr}(G) = k$ if and only if there is some $M \in \mathcal{MR}(H)$ such that every edge and vertex in $G - H$ is rank-preserving with respect to M. Conversely, $\mathrm{mr}(G) > k$ if and only if there is some set of edges $E' \subseteq E(G - H)$ and vertices $V' \subseteq V(G - H)$ such that $\mathcal{I}_{E'} \cup \mathcal{I}_{V'} = \mathcal{MR}(H)$.

Although the inclusion of a vertex set in the second statement is only necessary in the case when $|G - H| = 1$, it will become apparent that the additional flexibility it introduces enables us to manage quite a large number of cases in the proof of our main result.

Corollary 28. *Assume that* $\mathrm{mr}(G) > k$. *If there are sets* $E' \subset E(G - H)$ *and* $V' \subset V(G - H)$ *such that* $\mathcal{I}_{E'} \cup \mathcal{I}_{V'} = \mathcal{MR}(H)$ *and* $(\cup_{xy \in E'} \{x, y\}) \cup V' \subset V(G - H)$ *is a proper subset of* $V(G - H)$, *then* $G \notin \mathcal{F}_{k+1}(H)$.

Proof. Let $v \in V(G-H)\backslash((\cup_{xy\in E'}\{x,y\})\cup V')$. Then $E' \subset E((G-v)-H)$ and $V' \subset V((G-v)-H)$, so $\mathrm{mr}(G-v) > k$ and $G \notin \mathcal{F}_{k+1}(H)$. ☐

Proposition 29. *Let $G \in \mathcal{F}_{k+1}(H)$. If $|G-H| \geq 2$, then for every vertex v in $G-H$, $\mathcal{I}_v \neq \mathcal{MR}(H)$. If $|G-H| \geq 3$, then for every edge uv in $G-H$, $\mathcal{I}_{uv} \neq \mathcal{MR}(H)$.*

Proof. Suppose that $G \in \mathcal{F}_{k+1}(H)$. Suppose there is some vertex $v \in G-H$ which is rank-increasing with respect to every $M \in \mathcal{MR}(H)$. Let w be a vertex in $G-H$ other than v. Then $\mathrm{mr}(G-w) > k$, which is a contradiction.

Similarly, suppose that $G \in \mathcal{F}_{k+1}(H)$. Suppose there is some edge uv in $G-H$ which is rank-increasing with respect to every $M \in \mathcal{MR}(H)$. Let w be a vertex in $G-H$ other than u or v. Then $\mathrm{mr}(G-w) > k$, which is a contradiction. ☐

Corollary 30. *Let $G \in \mathcal{F}_{k+1}(H)$ and suppose that $|\mathcal{MR}(H)| = 1$. If $|G-H| \geq 2$, then $\mathcal{I}_v = \emptyset$ for every vertex v in $G-H$. If $|G-H| \geq 3$, then $\mathcal{I}_{uv} = \emptyset$ for every edge uv in $G-H$.*

Corollary 31. *Suppose that $|\mathcal{MR}(H)| = 1$. If $G \in \mathcal{F}_{k+1}(H)$, then $|G-H| \leq 2$.*

Proof. Suppose that $|\mathcal{MR}(H)| = 1$ and $\mathcal{MR}(H) = \{M\}$. Then if $|G-H| \geq 3$, $\mathcal{I}_{uv} = \emptyset$ for every edge uv in $G-H$ and $\mathcal{I}_v = \emptyset$ for every vertex v in $G-H$. Since every edge and vertex in $G-H$ is rank-preserving with respect to M, $\mathrm{mr}(G) = \mathrm{mr}(H) = k$ and $G \notin \mathcal{F}_{k+1}(H)$. ☐

Corollary 32. *Let $G \in \mathcal{F}_{k+1}(H)$. If $|G-H| \geq 3$, then $G-H$ contains no vertex v with $\mathrm{wt}(v) = \vec{0}$.*

Proof. Let $|G-H| \geq 3$ and let v be a vertex of $G-H$ with $\mathrm{wt}(v) = \vec{0}$, the zero vector. Suppose that there is some vertex w of $G-H$ distinct from v such that the edge vw has nonzero weight. Then the edge vw is rank-increasing for each $M \in \mathcal{MR}(H)$, so $\mathcal{I}_{vw} = \mathcal{MR}(H)$. This contradicts Proposition 29. Therefore, $\mathrm{wt}(vw) = 0$ for every $w \in V(G-H)$ and v is an isolated vertex in G. Therefore $\mathrm{mr}(G) = \mathrm{mr}(G-v) = k$, so $G \notin \mathcal{F}_{k+1}(H)$, a contradiction. ☐

Lemma 33. *Let $G \in \mathcal{F}_{k+1}(H)$. If $|G-H| \geq |\mathcal{C}(H)| + 1$, then $\mathcal{I}_{V(G-H)} \neq \mathcal{MR}(H)$ (i.e., there exists some $C \in \mathcal{C}$ for which $V(G-H)$ is rank-preserving with respect to each $M \in C$).*

Proof. Suppose that $\mathcal{I}_{V(G-H)} = \mathcal{MR}(H)$. Choose vertices $t_1, \ldots, t_{|\mathcal{C}(H)|}$ from $V(G-H)$ such that $C_i \subseteq \mathcal{I}_{t_i}$ for $i = 1, \ldots, |\mathcal{C}(H)|$. Let T be the set containing $t_1, \ldots, t_{|\mathcal{C}(H)|}$. Then $|T| \leq |\mathcal{C}(H)|$ and $\mathcal{I}_T = \mathcal{MR}(H)$. Let $v \in V(G-H)\backslash T$. Then $\mathcal{I}_{V(G-H)\backslash\{v\}} = \mathcal{MR}(H)$ and $\mathrm{mr}(G-v) > k$, which is a contradiction. Thus there is some $M \in \mathcal{MR}(H)$ and corresponding $C \in \mathcal{C}(H)$ for which $V(G-H)$ is rank-preserving. ☐

By Observation 27, $\mathrm{mr}(G) > k$ if and only if there exist subsets $E' \subseteq E(G-H)$ and $V' \subseteq V(G-H)$ such that $\mathcal{I}_{E'} \cup \mathcal{I}_{V'} = \mathcal{MR}(H)$. We will be interested in "minimal" subsets $R \subseteq E(G-H)$ and $T \subseteq V(G-H)$ such that $\mathcal{I}_R \cup \mathcal{I}_T = \mathcal{MR}(H)$ because R and T provide an upper bound for $|G-H|$, as the following theorem shows.

Theorem 34. *Assume that $\mathrm{mr}(G) > k$. Let R be a set of edges in $G-H$ and T be a set of vertices in $G-H$ such that $\mathcal{I}_R \cup \mathcal{I}_T = \mathcal{MR}(H)$. Let $S = \cup_{ij\in R}\{i,j\}$, the set of vertices incident to the edges in R. If $G \in \mathcal{F}_{k+1}(H)$, then $|G-H| \leq |S| + |T| \leq 2|R| + |T|$.*

Proof. We prove the contrapositive. Suppose that $|G - H| > |S| + |T|$ for some R, S, and T satisfying the hypotheses. Let $v \in V(G - H) \setminus (S \cup T)$ be a vertex in $G - H$ that is different from the vertices in S or T. Then $\text{mr}(G - v) > k$ and $G \notin \mathcal{F}_{k+1}(H)$. $\square$

The basic idea behind our strategy is to minimize the size of $|S| + |T|$ to get an upper bound on the number of vertices in $G - H$ for which $G \in \mathcal{F}_{k+1}(H)$.

In our proofs in Sections 4–9, we will examine possible cases for $\mathcal{I}_S$, $\mathcal{I}_R$, and $\mathcal{I}_T$. The following four properties will significantly reduce the number of cases we will need to consider.

Assume that G is a graph such that $\text{mr}(G) > k$. Let $R \subseteq E(G - H)$ and $T \subseteq V(G - H)$. Let $S = \cup_{ij \in R}\{i, j\}$, the set of vertices incident to the edges in R. Then the following properties are a direct consequence of the definition of rank-increasing vertices and edges.

P1. $\mathcal{I}_S$ and $\mathcal{I}_T$ are each the union of equivalence classes in $\mathcal{C}(H)$.

P2. $\mathcal{I}_S \subseteq \mathcal{I}_R$ since if $v \in S$ is rank-increasing for a matrix $M \in \mathcal{MR}(H)$, then any edge incident to v is also rank-increasing for M (Observation 20).

In addition, if $G \in \mathcal{F}_{k+1}(H)$, $|G - H| \geq |\mathcal{C}(H)| + 1$, and $\mathcal{I}_R \cup \mathcal{I}_T = \mathcal{MR}(H)$, the following properties are consequences of Lemma 33.

P3. $\bar{\mathcal{I}}_S \cup \bar{\mathcal{I}}_T \neq \mathcal{C}(H)$. This implies that $\mathcal{I}_S \neq \mathcal{MR}(H)$ and $\mathcal{I}_T \neq \mathcal{MR}(H)$.

P4. There exists a $C \in \mathcal{C}(H)$ such that $C \subseteq \mathcal{I}_R \setminus \mathcal{I}_S$. This implies that $\mathcal{I}_R \neq \emptyset$.

Property 4 is a consequence of $\mathcal{I}_R \cup \mathcal{I}_T = \mathcal{MR}(H)$ and properties P1 and P3.

Definition 35. Assume that $\text{mr}(G) > k$. Let $\mathcal{A}$ be the set of triples (R, S, T) such that

(a) $R \subseteq E(G - H)$, $S = \cup_{ij \in R}\{i, j\}$, and $T \subseteq V(G - H)$;

(b) $\mathcal{I}_R \cup \mathcal{I}_T = \mathcal{MR}(H)$; and

(c) $2|R| + |T|$ is minimized.

From the triples in $\mathcal{A}$, select those that minimize $|R|$, and from these triples, choose the triples (R, S, T) that minimize $|S|$. We call such an (R, S, T) an *optimal triple* for G and H.

Theorem 36. *Assume that* $\text{mr}(G) > k$. *Let* (R, S, T) *be an optimal triple for* G *and* H. *Then*

(a) For every $v \in T$, $\mathcal{I}_v \not\subseteq (\mathcal{I}_{T \setminus \{v\}} \cup \mathcal{I}_S)$, *and*

(b) For every $uv \in R$, $\mathcal{I}_{uv} \not\subseteq (\mathcal{I}_{R \setminus \{uv\}} \cup \mathcal{I}_S \cup \mathcal{I}_T)$.

Proof. Suppose that S and T do not satisfy (a). Let v be a vertex for which the property does not hold. Then $\mathcal{I}_R \cup \mathcal{I}_{T \setminus \{v\}} = \mathcal{MR}(H)$, but $2|R| + |T \setminus \{v\}| < 2|R| + |T|$. This is a contradiction since $(R, S, T) \in \mathcal{A}$.

Suppose that R, S, and T do not satisfy (b). Let uv be an edge for which the property does not hold. Let $R' = R \setminus \{uv\}$, $S' = \cup_{xy \in R'}\{x, y\}$, and $T' = T \cup \{u, v\}$. Then $\mathcal{I}_{R'} \cup \mathcal{I}_{T'} = \mathcal{MR}(H)$ and $2|R'| + |T'| \leq 2|R| + |T|$, so $(R', S', T') \in \mathcal{A}$. However, $|R'| < |R|$. This is a contradiction since (R, S, T) is an optimal triple. $\square$

The minimality of $|S|$ was not used in the proof of Theorem 36, but will be used later.

Let (R, S, T) be an optimal triple for G and H. Theorem 36(a) implies that for every vertex $v \in T$, there is class of matrices $C \in \mathcal{C}(H)$ such that v is rank-increasing with respect to every matrix in C, while every other vertex in T and every vertex in S is rank-preserving with respect to every matrix in C. Consequently, there are at most $|\bar{\mathcal{I}}_T \setminus \bar{\mathcal{I}}_S|$ vertices in T. Theorem 36(b) implies that for every edge $uv \in R$, there is some matrix $M \in \mathcal{MR}(H)$ such that uv is rank-increasing with respect to M, while every other edge in R and every vertex in $S \cup T$ is rank-preserving with respect to M. Consequently, there are at most $|\mathcal{I}_R \setminus \mathcal{I}_{S \cup T}| = |\mathcal{I}_R \setminus (\mathcal{I}_S \cup \mathcal{I}_T)|$ edges in R.

Corollary 37. *Assume that* $\operatorname{mr}(G) > k$. *If (R, S, T) is an optimal triple for G and H, then*

(a) $|T| \leq |\bar{\mathcal{I}}_T \setminus \bar{\mathcal{I}}_S| = |\{C \in \mathcal{C}(H) \mid C \subseteq (\mathcal{I}_T \setminus \mathcal{I}_S)\}|$, *and*

(b) $|R| \leq |\mathcal{I}_R \setminus (\mathcal{I}_S \cup \mathcal{I}_T)|$.

This corollary gives one upper bound for $|R|$. There will be times that we can prove that an edge in R is rank-increasing for one matrix $M_i \in \mathcal{MR}(H)$ if and only if it is also rank-increasing for another matrix $M_j \in \mathcal{MR}(H)$. In these cases, we can get a smaller upper bound for $|R|$.

Corollary 38. *Assume that* $\operatorname{mr}(G) > k$ *and let (R, S, T) be an optimal triple for G and H. Then $S \cap T = \emptyset$.*

Corollary 39. *Let $G \in \mathcal{F}_{k+1}(H)$ and let (R, S, T) be an optimal triple for G and H. If $\mathcal{I}_R = \mathcal{MR}(H)$, then $T = \emptyset$ and $|G - H| \leq |S| \leq 2|R|$.*

Proof. Since $\mathcal{I}_R = \mathcal{MR}(H)$, $\mathcal{I}_R \cup \mathcal{I}_\emptyset = \mathcal{MR}(H)$. Since for any $T \subseteq V(G-H)$, $2|R| + |\emptyset| \leq 2|R| + |T|$, we have $T = \emptyset$ by the minimality of $2|R| + |T|$. By Theorem 34, $|G - H| \leq |S|$. ☐

The following lemma and corollary give conditions sufficient to reduce the size of the upper bound for $|S|$.

Lemma 40. *Assume that* $\operatorname{mr}(G) > k$. *Let (R, S, T) be an optimal triple for G and H. Suppose that*

(a) $|R| = 2$,

(b) If uv and wx are any two edges between vertices in S, then either $\mathcal{I}_{wx} \setminus \mathcal{I}_S = \mathcal{I}_{uv} \setminus \mathcal{I}_S$ or $\mathcal{I}_{wx} \setminus \mathcal{I}_S = (\mathcal{I}_R \setminus \mathcal{I}_{uv}) \setminus \mathcal{I}_S$, and

(c) there are two (not necessarily distinct) vertices v and w, one incident to each edge of R, such that $\mathcal{I}_{\{v,w\}} = \mathcal{I}_S$.

Then $|S| = 3$.

Proof. Since $|R| = 2$, we have $3 \leq |S| \leq 4$. Suppose that $|S| = 4$. Let $R = \{uv, wx\}$ and $S = \{u, v, w, x\}$, where $\mathcal{I}_{\{v,w\}} = \mathcal{I}_S$. Let $A = \mathcal{I}_{uv} \setminus \mathcal{I}_S$ and $B = (\mathcal{I}_R \setminus \mathcal{I}_{uv}) \setminus \mathcal{I}_S$. We have $\mathcal{I}_{wx} \setminus \mathcal{I}_S \neq A$ by Theorem 36(b), so $\mathcal{I}_{wx} \setminus \mathcal{I}_S = B$ by hypothesis (b). By hypothesis (b), $\mathcal{I}_{vw} \setminus \mathcal{I}_S = A$ or $\mathcal{I}_{vw} \setminus \mathcal{I}_S = B$.

I. $\mathcal{I}_{vw} \setminus \mathcal{I}_S = A$. Let $R' = \{vw, wx\}$ and $S' = \{v, w, x\}$.

II. $\mathcal{I}_{vw} \setminus \mathcal{I}_S = B$. Let $R' = \{uv, vw\}$ and $S' = \{u, v, w\}$.

Since $\{v, w\} \subseteq S'$, $\mathcal{I}_{S'} = \mathcal{I}_S$. Also $\mathcal{I}_{R'} = A \cup B \cup \mathcal{I}_{S'} = A \cup B \cup \mathcal{I}_S = \mathcal{I}_R$. Therefore, (R', S', T) is a triple such that $\mathcal{I}_{R'} \cup \mathcal{I}_T = \mathcal{MR}(H)$, $2|R'| + |T| = 2|R| + |T|$, and $|R'| = |R|$, but $|S'| < |S|$, which contradicts the optimality of (R, S, T). Thus $|S| = 3$. □

Corollary 41. *Assume that* $\mathrm{mr}(G) > k$. *Let* (R, S, T) *be an optimal triple for* G *and* H. *Suppose that*

(a) $|R| = 2$,

(b) If uv *and* wx *are any two edges between vertices in* S, *then either* $\mathcal{I}_{wx} \setminus \mathcal{I}_S = \mathcal{I}_{uv} \setminus \mathcal{I}_S$ *or* $\mathcal{I}_{wx} \setminus \mathcal{I}_S = (\mathcal{I}_R \setminus \mathcal{I}_{uv}) \setminus \mathcal{I}_S$, *and*

(c) $|\bar{\mathcal{I}}_S| \leq 1$.

Then $|S| = 3$.

Proof. Since $|\bar{\mathcal{I}}_S| \leq 1$, there is some vertex $y \in S$ such that $\mathcal{I}_y = \mathcal{I}_S$. Therefore $\mathcal{I}_{\{y,z\}} = \mathcal{I}_S$ for any vertex $z \in S$. Applying Lemma 40 then gives the result. □

In Sections 4–9, we will determine an upper bound for the number of vertices in graphs in $\mathcal{F}_4(H)$ for each graph H in

$$\mathcal{F}_3 = \{3K_2, P_3 \vee P_3, \text{dart}, \ltimes, P_3 \cup K_2, \text{full house}, P_4\}.$$

We will then apply Corollary 14 to determine the maximum number of vertices in a graph in $\mathcal{F}_4$.

4 $H = 3K_2$ or $H = P_3 \vee P_3$

By Proposition 17, $|\mathcal{MR}(3K_2)| = 1$ and $|\mathcal{MR}(P_3 \vee P_3)| = 1$, so applying Corollary 31 gives the following lemma.

Lemma 42. *If* $G \in \mathcal{F}_4(3K_2)$ *or* $G \in \mathcal{F}_4(P_3 \vee P_3)$, *then* $|G| \leq 8$.

5 $H =$ Dart

Lemma 43. *If* $G \in \mathcal{F}_4(\text{dart})$, *then* $|G| \leq 7$.

Proof. Suppose that $G \in \mathcal{F}_4(\text{dart})$ and $|G| \geq 8$ (i.e., $|G - H| \geq 3$). Then $G - H$ has no vertices with zero weight by Corollary 32. Assume that (R, S, T) is an optimal triple for G and the dart. Let $\mathcal{MR}(\text{dart}) = \{M_1, M_2\}$ and $\mathcal{C}(\text{dart}) = \{C_1 = \{M_1\}, C_2 = \{M_2\}\}$ be as in Proposition 17(c). By property P1, $\mathcal{I}_S \in \{\emptyset, C_1, C_2, C_1 \cup C_2\}$. By property P3, $\mathcal{I}_S \neq C_1 \cup C_2$. Thus $\mathcal{I}_S \in \{\emptyset, C_1, C_2\}$ and we have the following cases.

Case 1: $\mathcal{I}_S = \emptyset$**.** By Observation 26, if $v \in S$, then

$$\mathrm{wt}(v) \in \mathrm{col}(M_1) \cap \mathrm{col}(M_2) = \{\vec{0},\ v_1 = (0,1,0,1,0)^T,\ v_2 = (1,0,0,1,0)^T,\ v_3 = (1,1,0,0,0)^T\}.$$

The rank-preserving tables for $[v_1, v_2, v_3]$ with respect to M_1 and M_2 are, respectively,

$$P_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad P_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Since $P_1 = P_2$, an edge in R is rank-preserving for M_1 if and only if it is also rank-preserving for M_2. This combined with property P4 implies that $\mathcal{I}_R = \{M_1, M_2\}$. Since $M_1 \in \mathcal{I}_{uv}$ if and only if $M_2 \in \mathcal{I}_{uv}$ for any edge $uv \in R$, Theorem 36(b) implies that $|R| = 1$ and $|S| = 2$. Since $\mathcal{I}_R = \mathcal{MR}(\text{dart})$, $T = \emptyset$ and $|G - H| \leq 2$ by Corollary 39. This contradicts our assumption that $|G| \geq 8$, so this case cannot occur.

Case 2: $\mathcal{I}_S = \{M_1\}$ or $\mathcal{I}_S = \{M_2\}$. In each of these cases, by property P4, $\mathcal{I}_R = \{M_1, M_2\}$. By Corollary 37, $|R| \leq 1$, so $|R| = 1$. Again, since $\mathcal{I}_R = \mathcal{MR}(\text{dart})$, $T = \emptyset$ and $|G - H| \leq 2$ by Corollary 39. This contradicts our assumption that $|G| \geq 8$, so neither of these cases can occur.

Thus $|G - H| \geq 3$ is impossible, so $|G - H| \leq 2$ and $|G| \leq 7$. □

6 $H = \ltimes$

Lemma 44. *If $G \in \mathcal{F}_4(\ltimes)$, then $|G| \leq 8$.*

Proof. Suppose that $G \in \mathcal{F}_4(\ltimes)$ and $|G| \geq 8$ (i.e., $|G - H| \geq 3$). Then $G - H$ has no vertices with zero weight by Corollary 32. Assume that (R, S, T) is an optimal triple for G and $\ltimes$. Let $\mathcal{MR}(\ltimes) = \{M_1, M_2, M_3\}$ and $\mathcal{C}(\ltimes) = \{C_1 = \{M_1, M_2\}, C_2 = \{M_3\}\}$ be as in Proposition 17(d). By property P1, $\mathcal{I}_S \in \{\emptyset, C_1, C_2, C_1 \cup C_2\}$. By property P3, $\mathcal{I}_S \neq C_1 \cup C_2$. Thus $\mathcal{I}_S \in \{\emptyset, C_1, C_2\}$ and we have the following cases.

Case 1: $\mathcal{I}_S = \emptyset$. By Observation 26, if $v \in S$, then

$$\text{wt}(v) \in \bigcap_{i=1}^{3} \text{col}(M_i) = \{\vec{0},\, v_1 = (0,1,1,0,0)^T,\, v_2 = (1,0,0,1,1)^T,\, v_3 = (1,1,1,1,1)^T\}.$$

The rank-preserving tables for $[v_1, v_2, v_3]$ with respect to M_1, M_2, and M_3 are, respectively,

$$P_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad \text{and } P_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Since $P_2 = P_3$, an edge in R is rank-preserving for M_2 if and only if it is also rank-preserving for M_3. Thus, we must either have both M_2 and M_3 in $\mathcal{I}_R$ or have neither in the set. Thus $\mathcal{I}_R \in \{\{M_1, M_2, M_3\}, \{M_2, M_3\}, \{M_1\}\}$. By property P4, $\mathcal{I}_R \neq \{M_1\}$. Therefore we have the following cases.

Subcase 1.1: $\mathcal{I}_R = \{M_1, M_2, M_3\}$. Since $\mathcal{I}_R = \mathcal{MR}(\ltimes)$, Corollary 39 implies that $T = \emptyset$ and $|G - H| \leq |S| \leq 2|R|$. Since $M_2 \in \mathcal{I}_{uv}$ if and only if $M_3 \in \mathcal{I}_{uv}$ for each edge $uv \in R$, $|R| \leq 2$ by Theorem 36(b). If $|R| \leq 1$, then $|G - H| \leq 2$, a contradiction. If $|R| = 2$, then Theorem 36(b) implies that R consists of an edge uv such that $\mathcal{I}_{uv} = \{M_1\}$ and another edge wx such that $\mathcal{I}_{wx} = \{M_2, M_3\}$. Since the second row and column of P_1, P_2, and P_3 are identical, we see that if any vertex in S, say u, has weight v_2, then the edge in R incident to the vertex must have either $\mathcal{I}_{uv} = \mathcal{I}_R$ or $\mathcal{I}_{uv} = \emptyset$. Neither of these cases occur, so u, v, w, and x each must have weight v_1 or v_3. Note that since the principal submatrices $P_1[1,3]$ and $P_2[1,3] = P_3[1,3]$ are complementary, any edge between vertices with weights v_1 or v_3 must be either rank-increasing for M_1 and rank-preserving for M_2 and M_3, or rank-increasing for M_2 and M_3 and rank-preserving for M_1. This fact combined with the facts that $|R| = 2$ and $|\bar{\mathcal{I}}_S| = 0$ allow us to apply Corollary 41 to conclude that $|S| = 3$, $|G - H| \leq 3$, and $|G| \leq 8$.

Subcase 1.2: $\mathcal{I}_R = \{M_2, M_3\}$**.** Since $\mathcal{I}_R \cup \mathcal{I}_T = \mathcal{MR}(H)$, properties P1 and P3 imply that $\mathcal{I}_T = \{M_1, M_2\}$. By Corollary 37, $|R| \leq 1$ and $|T| \leq 1$. By Theorem 34, $|G - H| \leq 3$, so $|G| \leq 8$.

Case 2: $\mathcal{I}_S = \{M_1, M_2\}$**.** By property P4, $\mathcal{I}_R = \{M_1, M_2, M_3\}$, so $T = \emptyset$ and $|G - H| \leq 2|R|$ by Corollary 39. By Corollary 37, $|R| \leq 1$, so $|G - H| \leq 2$. This contradicts the assumption that $|G - H| \geq 3$, so this case cannot occur.

Case 3: $\mathcal{I}_S = \{M_3\}$**.** By Observation 26, if $v \in S$, then

$$\begin{aligned}\mathrm{wt}(v) \in \mathrm{col}(M_1) = \mathrm{col}(M_2) = \{&\vec{0},\ v_1 = (0,0,0,1,1)^T,\ v_2 = (0,1,1,0,0)^T,\ v_3 = (0,1,1,1,1)^T,\\ &v_4 = (1,0,0,0,0)^T,\ v_5 = (1,0,0,1,1)^T,\ v_6 = (1,1,1,0,0)^T,\ v_7 = (1,1,1,1,1)^T\}.\end{aligned}$$

The rank-preserving tables for $[v_1, v_2, v_3, v_4, v_5, v_6, v_7]$ with respect to M_1 and M_2 are, respectively,

$$P_1 = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad P_2 = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

By property P4, $\mathcal{I}_R = \{M_1, M_2, M_3\}$, so $T = \emptyset$ and $|G - H| \leq |S|$ by Corollary 39. By Corollary 37, $|R| \leq 2$. If $|R| = 1$, then $|G - H| \leq 2$, which is a contradiction. If $|R| = 2$, then Theorem 36(b) implies that R consists of an edge uv such that $\mathcal{I}_{uv} \setminus \mathcal{I}_S = \{M_1\}$ and another edge wx such that $\mathcal{I}_{wx} \setminus \mathcal{I}_S = \{M_2\}$. Since the first, fourth, and fifth rows and columns of P_1 and P_2 are identical, we see that if any vertex, say u, has weight v_1, v_4, or v_5, then the edge in R incident to the vertex must have either $\mathcal{I}_{uv} \setminus \mathcal{I}_S = \mathcal{I}_R \setminus \mathcal{I}_S$ or $\mathcal{I}_{uv} \setminus \mathcal{I}_S = \emptyset$. Neither of these cases occur, so u, v, w, and x each must have weight v_2, v_3, v_6, or v_7. As in Subcase 1.1, since $P_1[2,3,6,7]$ and $P_2[2,3,6,7]$ are complementary, $|R| = 2$, and $|\bar{\mathcal{I}}_S| = 1$, we can apply Corollary 41 to conclude that $|S| = 3$, $|G - H| \leq 3$, and $|G| \leq 8$. ☐

7 $H = P_3 \cup K_2$

Lemma 45. *If $G \in \mathcal{F}_4(P_3 \cup K_2)$, then $|G| \leq 8$.*

Proof. Suppose that $G \in \mathcal{F}_4(P_3 \cup K_2)$ and $|G| \geq 8$ (i.e., $|G - H| \geq 3$). Then $G - H$ has no vertices with zero weight by Corollary 32. Assume that (R, S, T) is an optimal triple for G and $P_3 \cup K_2$. Let $\mathcal{MR}(P_3 \cup K_2) = \{M_1, M_2, M_3\}$ and $\mathcal{C}(P_3 \cup K_2) = \{C_1 = \{M_1, M_2\}, C_2 = \{M_3\}\}$ be as in Proposition 17(e). By properties P1 and P3, $\mathcal{I}_S \in \{\emptyset, C_1, C_2\}$, so we have the following cases.

Case 1: $\mathcal{I}_S = \emptyset$**.** By Observation 26, if $v \in S$, then

$$\mathrm{wt}(v) \in \bigcap_{i=1}^{3} \mathrm{col}(M_i) = \{\vec{0},\ v_1 = (0,0,0,1,1)^T,\ v_2 = (1,0,1,0,0)^T,\ v_3 = (1,0,1,1,1)^T\}.$$

The rank-preserving tables for $[v_1, v_2, v_3]$ with respect to M_1, M_2, and M_3 are, respectively,

$$P_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad \text{and } P_3 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Since $P_1 = P_3$, an edge in R is rank-preserving for M_1 if and only if it is also rank-preserving for M_3. Thus $\mathcal{I}_R \in \{\{M_1, M_2, M_3\}, \{M_1, M_3\}, \{M_2\}\}$. By property P4, $\mathcal{I}_R \neq \{M_2\}$. Therefore we have the following cases.

Subcase 1.1: $\mathcal{I}_R = \{M_1, M_2, M_3\}$**.** Since $\mathcal{I}_R = \mathcal{MR}(P_3 \cup K_2)$, Corollary 39 implies that $T = \emptyset$ and $|G - H| \leq |S| \leq 2|R|$. We reason as in Subcase 1.1 in Section 6. Since $M_1 \in \mathcal{I}_{uv}$ if and only if $M_3 \in \mathcal{I}_{uv}$ for each edge $uv \in R$, $|R| \leq 2$ by Theorem 36(b). If $|R| = 1$, then $|G - H| \leq 2$, a contradiction. If $|R| = 2$, then Theorem 36(b) implies that R consists of an edge uv such that $\mathcal{I}_{uv} = \{M_1, M_3\}$ and another edge wx such that $\mathcal{I}_{wx} = \{M_2\}$. Since the first row and column of P_1, P_2, and P_3 are identical, we see that if any vertex, say u, has weight v_1, then the edge in R incident to the vertex must have either $\mathcal{I}_{uv} = \mathcal{I}_R$ or $\mathcal{I}_{uv} = \emptyset$. Neither of these cases occur, so u, v, w, and x each must have weight v_2 or v_3. As in Subcase 1.1 in Section 6, since $P_1[2, 3]$ and $P_2[2, 3]$ are complementary, $|R| = 2$, and $|\bar{\mathcal{I}}_S| = 0$, we can apply Corollary 41 to conclude that $|S| = 3$, $|G - H| \leq 3$, and $|G| \leq 8$.

Subcase 1.2: $\mathcal{I}_R = \{M_1, M_3\}$**.** Since $\mathcal{I}_R \cup \mathcal{I}_T = \mathcal{MR}(H)$, properties P1 and P3 imply that $\mathcal{I}_T = \{M_1, M_2\}$. By Corollary 37, $|R| \leq 1$ and $|T| \leq 1$. By Theorem 34, $|G - H| \leq 3$, so $|G| \leq 8$.

Case 2: $\mathcal{I}_S = \{M_1, M_2\}$**.** By property P4, $\mathcal{I}_R = \{M_1, M_2, M_3\}$, so $T = \emptyset$ and $|G - H| \leq 2|R|$ by Corollary 39. By Corollary 37, $|R| \leq 1$, so $|G - H| \leq 2$ and $|G| \leq 7$. This contradicts the assumption that $|G| \geq 8$, so this case cannot occur.

Case 3: $\mathcal{I}_S = \{M_3\}$**.** By Observation 26, if $v \in S$, then

$$\begin{aligned} \text{wt}(v) \in \text{col}(M_1) = \text{col}(M_2) = \{&\vec{0},\, v_1 = (0,0,0,1,1)^T,\, v_2 = (0,1,0,0,0)^T,\, v_3 = (0,1,0,1,1)^T, \\ &v_4 = (1,0,1,0,0)^T,\, v_5 = (1,0,1,1,1)^T,\, v_6 = (1,1,1,0,0)^T,\, v_7 = (1,1,1,1,1)^T\}. \end{aligned}$$

The rank-preserving tables for $[v_1, v_2, v_3, v_4, v_5, v_6, v_7]$ with respect to M_1 and M_2 are, respectively,

$$P_1 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad P_2 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}.$$

By property P4, $\mathcal{I}_R = \{M_1, M_2, M_3\}$, so $T = \emptyset$ and $|G - H| \leq |S|$ by Corollary 39. By Corollary 37, $|R| \leq 2$. If $|R| = 1$, then $|G - H| \leq 2$, which is a contradiction. If $|R| = 2$, then Theorem 36(b) implies that R consists of an edge uv such that $\mathcal{I}_{uv} \setminus \mathcal{I}_S = \{M_1\}$ and another edge wx such that $\mathcal{I}_{wx} \setminus \mathcal{I}_S = \{M_2\}$. Since the first three rows and columns of P_1 and P_2 are identical, we see that if any vertex, say u, has weight v_1, v_2, or v_3, then the edge in R incident to the vertex must have either $\mathcal{I}_{uv} \setminus \mathcal{I}_S = \mathcal{I}_R \setminus \mathcal{I}_S$ or $\mathcal{I}_{uv} \setminus \mathcal{I}_S = \emptyset$. Neither of these cases occur, so u, v, w, and x each must have weight v_4, v_5, v_6, or v_7. As in Subcase 1.1 in Section 6, since $P_1[4, 5, 6, 7]$ and $P_2[4, 5, 6, 7]$ are complementary, $|R| = 2$, and $|\bar{\mathcal{I}}_S| = 1$, we can apply Corollary 41 to conclude that $|S| = 3$, $|G - H| \leq 3$, and $|G| \leq 8$. □

8 H = full house

Lemma 46. *If $G \in \mathcal{F}_4$(full house), then $|G| \leq 8$.*

Proof. Suppose that $G \in \mathcal{F}_4$(full house) and $|G| \geq 9$ (i.e., $|G - H| \geq 4$). Then $G - H$ has no vertices with zero weight by Corollary 32. Assume that (R, S, T) is an optimal triple for G and the full house. Let $\mathcal{MR}$(full house) $= \{M_1, M_2, M_3, M_4\}$ and $\mathcal{C}$(full house) $= \{C_1 = \{M_1, M_2\}, C_2 = \{M_3\}, C_3 = \{M_4\}\}$ be as in Proposition 17(f). By properties P1 and P3, $\mathcal{I}_S \in \{\emptyset, C_1, C_2, C_3, C_1 \cup C_2, C_1 \cup C_3, C_2 \cup C_3\}$, so we have the following cases.

Case 1: $\mathcal{I}_S = \emptyset$. By Observation 26, if $v \in S$, then

$$\text{wt}(v) \in \bigcap_{i=1}^{4} \text{col}(M_i) = \{\vec{0},\, v_1 = (0,0,0,1,1)^T\}.$$

The rank-preserving tables for $[v_1]$ with respect to M_1, M_2, M_3, and M_4 are, respectively,

$$P_1 = \begin{bmatrix} 0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 1 \end{bmatrix}, \quad P_3 = \begin{bmatrix} 1 \end{bmatrix}, \quad \text{and } P_4 = \begin{bmatrix} 0 \end{bmatrix}.$$

Since $P_1 = P_4$ and $P_2 = P_3$ are complementary, an edge uv in R has either $\mathcal{I}_{uv} = \{M_1, M_4\}$ or $\mathcal{I}_{uv} = \{M_2, M_3\}$. Thus $\mathcal{I}_R \in \{\{M_1, M_2, M_3, M_4\}, \{M_1, M_4\}, \{M_2, M_3\}\}$. Therefore we have the following cases.

Subcase 1.1: $\mathcal{I}_R = \{M_1, M_2, M_3, M_4\}$. Since $\mathcal{I}_R = \mathcal{MR}$(full house), Corollary 39 implies that $T = \emptyset$ and $|G - H| \leq |S| \leq 2|R|$. We reason as in Subcase 1.1 in Section 6. Since $M_1 \in \mathcal{I}_{uv}$ if and only if $M_4 \in \mathcal{I}_{uv}$ and $M_2 \in \mathcal{I}_{uv}$ if and only if $M_3 \in \mathcal{I}_{uv}$ for each edge $uv \in R$, $|R| \leq 2$ by Theorem 36(b). If $|R| = 1$, then $|G - H| \leq 2$, a contradiction. If $|R| = 2$, then Theorem 36(b) implies that R consists of an edge uv such that $\mathcal{I}_{uv} = \{M_1, M_4\}$ and another edge wx such that $\mathcal{I}_{wx} = \{M_2, M_3\}$. As in Subcase 1.1 in Section 6, since $P_1 = P_4$ and $P_2 = P_3$ are complementary, $|R| = 2$, and $|\bar{\mathcal{I}}_S| = 0$, we can apply Corollary 41 to conclude that $|S| = 3$, $|G - H| \leq 3$, and $|G| \leq 8$. This contradicts the assumption that $|G| \geq 9$, so this case does not occur.

Subcase 1.2: $\mathcal{I}_R = \{M_1, M_4\}$. Since $\mathcal{I}_R \cup \mathcal{I}_T = \mathcal{MR}(H)$, properties P1 and P3 imply that $\mathcal{I}_T = \{M_1, M_2, M_3\}$. By Corollary 37, $|R| \leq 1$ and $|T| \leq 2$. If $|T| = 2$, then by Theorem 36(a), T consists of a vertex v such that $\mathcal{I}_v = \{M_1, M_2\}$ and another vertex w such that $\mathcal{I}_w = \{M_3\}$. Since v is rank-preserving with respect to M_3 and M_4, $\text{wt}(v) \in \text{col}(M_3) \cap \text{col}(M_4) = \{\vec{0},\, (0,0,0,1,1)^T\}$, so $\text{wt}(v) = (0,0,0,1,1)^T$. But then $\mathcal{I}_v = \emptyset$, a contradiction, so this case does not occur.

Subcase 1.3: $\mathcal{I}_R = \{M_2, M_3\}$. Since $\mathcal{I}_R \cup \mathcal{I}_T = \mathcal{MR}(H)$, properties P1 and P3 imply that $\mathcal{I}_T = \{M_1, M_2, M_4\}$. By Corollary 37, $|R| \leq 1$ and $|T| \leq 2$. Again, if $|T| = 2$, then by Theorem 36(a), T consists of a vertex v such that $\mathcal{I}_v = \{M_1, M_2\}$ and another vertex w such that $\mathcal{I}_w = \{M_4\}$. Proceeding as in Subcase 1.2, $\text{wt}(v) = (0,0,0,1,1)^T$ and $\mathcal{I}_v = \emptyset$, a contradiction, so this case does not occur.

Case 2: $\mathcal{I}_S = \{M_1, M_2\}$. By Observation 26, if $v \in S$, then $\text{wt}(v) \in \text{col}(M_3) \cap \text{col}(M_4) = \{\vec{0},\, v_1 = (0,0,0,1,1)^T\}$. But then $\mathcal{I}_S = \emptyset$, a contradiction, so this case does not occur.

Case 3: $\mathcal{I}_S = \{M_3\}$**.** By Observation 26, if $v \in S$, then

$$\mathrm{wt}(v) \in \mathrm{col}(M_1) \cap \mathrm{col}(M_2) \cap \mathrm{col}(M_4) = \{\vec{0},\ v_1 = (0,0,0,1,1)^T,\ v_2 = (1,1,1,0,0)^T,\ v_3 = (1,1,1,1,1)^T\}.$$

The rank-preserving tables for $[v_1, v_2, v_3]$ with respect to M_1, M_2, and M_4 are, respectively,

$$P_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad \text{and } P_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Since $P_1 = P_4$, an edge in R is rank-preserving for M_1 if and only if it is also rank-preserving for M_4. Thus $\mathcal{I}_R \in \{\{M_1, M_2, M_3, M_4\}, \{M_1, M_3, M_4\}, \{M_2, M_3\}\}$. By property P4, $\mathcal{I}_R \neq \{M_2, M_3\}$. Therefore we have the following cases.

Subcase 3.1: $\mathcal{I}_R = \{M_1, M_2, M_3, M_4\}$**.** Since $\mathcal{I}_R = \mathcal{MR}$(full house), Corollary 39 implies that $T = \emptyset$ and $|G - H| \leq |S|$. We reason as in Subcase 1.1 in Section 6. Since $M_1 \in \mathcal{I}_{uv}$ if and only if $M_4 \in \mathcal{I}_{uv}$ for each edge $uv \in R$, $|R| \leq 2$ by Theorem 36(b). If $|R| = 1$, then $|G - H| \leq 2$, which is a contradiction. If $|R| = 2$, then Theorem 36(b) implies that R consists of an edge uv such that $\mathcal{I}_{uv} \setminus \mathcal{I}_S = \{M_1, M_4\}$ and another edge wx such that $\mathcal{I}_{wx} \setminus \mathcal{I}_S = \{M_2\}$. Since the third row and column of P_1, P_2, and P_4 are identical, we see that if any vertex, say u, has weight v_3, then the edge in R incident to the vertex must have either $\mathcal{I}_{uv} \setminus \mathcal{I}_S = \mathcal{I}_R \setminus \mathcal{I}_S$ or $\mathcal{I}_{uv} \setminus \mathcal{I}_S = \emptyset$. Neither of these cases occur, so u, v, w, and x each must have weight v_1 or v_2. As in Subcase 1.1 in Section 6, since $P_1[1,2] = P_4[1,2]$ and $P_2[1,2]$ are complementary, $|R| = 2$, and $|\bar{\mathcal{I}}_S| = 1$, we can apply Corollary 41 to conclude that $|S| = 3$, $|G - H| \leq 3$, and $|G| \leq 8$. This contradicts the assumption that $|G| \geq 9$, so this case does not occur.

Subcase 3.2: $\mathcal{I}_R = \{M_1, M_3, M_4\}$**.** Since $\mathcal{I}_R \cup \mathcal{I}_T = \mathcal{MR}(H)$, properties P1 and P3 imply that $\mathcal{I}_T = \{M_1, M_2\}$ or $\mathcal{I}_T = \{M_1, M_2, M_3\}$. In each of these cases, $|R| \leq 1$ and $|T| \leq 1$ by Theorem 36(b) and Corollary 37, implying that $|G - H| \leq 3$ and $|G| \leq 8$. This contradicts the assumption that $|G| \geq 9$, so these cases do not occur.

Case 4: $\mathcal{I}_S = \{M_4\}$**.** By Observation 26, if $v \in S$, then

$$\mathrm{wt}(v) \in \mathrm{col}(M_1) \cap \mathrm{col}(M_2) \cap \mathrm{col}(M_3) = \{\vec{0},\ v_1 = (0,0,0,1,1)^T,\ v_2 = (0,1,1,0,0)^T,\ v_3 = (0,1,1,1,1)^T\}.$$

The rank-preserving tables for $[v_1, v_2, v_3]$ with respect to M_1, M_2, and M_3 are, respectively,

$$P_1 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad \text{and } P_3 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Since $P_2 = P_3$, an edge in R is rank-preserving for M_2 if and only if it is also rank-preserving for M_3. By properties P2 and P4, $\mathcal{I}_R \in \{\{M_1, M_2, M_3, M_4\}, \{M_2, M_3, M_4\}\}$, so we have the following cases.

Subcase 4.1: $\mathcal{I}_R = \{M_1, M_2, M_3, M_4\}$**.** Since $\mathcal{I}_R = \mathcal{MR}(\text{full house})$, Corollary 39 implies that $T = \emptyset$ and $|G - H| \leq |S|$. We again reason as in Subcase 1.1 in Section 6. Since $M_2 \in \mathcal{I}_{uv}$ if and only if $M_3 \in \mathcal{I}_{uv}$ for each edge $uv \in R$, $|R| \leq 2$ by Theorem 36(b). If $|R| = 1$, then $|G - H| \leq 2$, which is a contradiction. If $|R| = 2$, then Theorem 36(b) implies that R consists of an edge uv such that $\mathcal{I}_{uv} \setminus \mathcal{I}_S = \{M_2, M_3\}$ and another edge wx such that $\mathcal{I}_{wx} \setminus \mathcal{I}_S = \{M_1\}$. Since the third row and column of P_1, P_2, and P_3 are identical, we see that if any vertex, say u, has weight v_3, then the edge in R incident to the vertex must have either $\mathcal{I}_{uv} \setminus \mathcal{I}_S = \mathcal{I}_R \setminus \mathcal{I}_S$ or $\mathcal{I}_{uv} \setminus \mathcal{I}_S = \emptyset$. Neither of these cases occur, so u, v, w, and x each must have weight v_1 or v_2. As in Subcase 1.1 in Section 6, since $P_1[1, 2]$ and $P_2[1, 2] = P_3[1, 2]$ are complementary, $|R| = 2$, and $|\bar{\mathcal{I}}_S| = 1$, we can apply Corollary 41 to conclude that $|S| = 3$, $|G - H| \leq 3$, and $|G| \leq 8$. This contradicts the assumption that $|G| \geq 9$, so this case does not occur.

Subcase 4.2: $\mathcal{I}_R = \{M_2, M_3, M_4\}$**.** Since $\mathcal{I}_R \cup \mathcal{I}_T = \mathcal{MR}(H)$, properties P1 and P3 imply that $\mathcal{I}_T = \{M_1, M_2\}$ or $\mathcal{I}_T = \{M_1, M_2, M_4\}$. In each of these cases, $|R| \leq 1$ and $|T| \leq 1$ by Theorem 36(b) and Corollary 37, implying that $|G - H| \leq 3$ and $|G| \leq 8$. This contradicts the assumption that $|G| \geq 9$, so these cases do not occur.

Case 5: $\mathcal{I}_S = \{M_1, M_2, M_3\}$ **or** $\mathcal{I}_S = \{M_1, M_2, M_4\}$**.** In each of these cases, by property P4, $\mathcal{I}_R = \{M_1, M_2, M_3, M_4\}$, so $T = \emptyset$ and $|G - H| \leq 2|R|$ by Corollary 39. In each of these cases, $|R| \leq 1$ by Corollary 37, so $|G - H| \leq 2$ and $|G| \leq 7$. This contradicts the assumption that $|G| \geq 9$, so these cases do not occur.

Case 6: $\mathcal{I}_S = \{M_3, M_4\}$**.** By Observation 26, if $v \in S$, then

$$\begin{aligned}\text{wt}(v) \in \text{col}(M_1) = \text{col}(M_2) = \{&\vec{0},\, v_1 = (0,0,0,1,1),\, v_2 = (0,1,1,0,0),\, v_3 = (0,1,1,1,1),\\ &v_4 = (1,0,0,0,0),\, v_5 = (1,0,0,1,1),\, v_6 = (1,1,1,0,0),\, v_7 = (1,1,1,1,1)\}.\end{aligned}$$

The rank-preserving tables for $[v_1, v_2, v_3, v_4, v_5, v_6, v_7]$ with respect to M_1 and M_2 are, respectively,

$$P_1 = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad P_2 = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

By property P4, $\mathcal{I}_R = \{M_1, M_2, M_3, M_4\}$, so $T = \emptyset$ and $|G - H| \leq |S| \leq 2|R|$ by Corollary 39. By Corollary 37, $|R| \leq 2$. If $|R| = 1$, then $|G - H| \leq 2$ and $|G| \leq 7$, a contradiction.

Suppose that $|R| = 2$. Let $R = \{uv, wx\}$. Theorem 36(b) implies that R consists of an edge e_1 such that $\mathcal{I}_{e_1} \setminus \mathcal{I}_S = \{M_1\}$ and another edge e_2 such that $\mathcal{I}_{e_2} \setminus \mathcal{I}_S = \{M_2\}$. Since the third, fourth, and seventh rows and columns of P_1 and P_2 are identical, we see that if any vertex, say a vertex in e_1, has weight v_3, v_4, or v_7, then the edge in R incident to the vertex must have either $\mathcal{I}_{e_1} \setminus \mathcal{I}_S = \mathcal{I}_R \setminus \mathcal{I}_S$ or $\mathcal{I}_{e_1} \setminus \mathcal{I}_S = \emptyset$. Neither of these cases occur, so each of the vertices in S must have weight v_1, v_2, v_5, or v_6. Note also that $P_1[1, 2, 5, 6]$ and $P_2[1, 2, 5, 6]$ are complementary. However, we cannot proceed as before and apply Corollary 41 since $|\bar{\mathcal{I}}_S| = 2$.

If there are vertices a and b, one incident to each edge of R, such that $\mathcal{I}_{\{a,b\}} = \mathcal{I}_S$, then we can apply Lemma 40 and conclude that $|S| = 3$, $|G - H| \leq 3$, and $|G| \leq 8$, a contradiction.

Suppose that $|S| = 4$ and there are not two vertices a and b in R such that a is incident to one edge, b is incident to the other edge, and $\mathcal{I}_{\{a,b\}} = \mathcal{I}_S = \{M_3, M_4\}$. By relabeling, if necessary, we then have $\mathcal{I}_u = \{M_3\}$, $\mathcal{I}_v = \{M_4\}$, $\mathcal{I}_w = \emptyset$, and $\mathcal{I}_x = \emptyset$. Recall also that for any vertex $a \in S$, $\mathrm{wt}(a) \in \{v_1, v_2, v_5, v_6\}$. Notice that if a vertex a has weight $\mathrm{wt}(a) = v_1$, then $\mathcal{I}_a = \emptyset$, so $\mathrm{wt}(u) \neq v_1$ and $\mathrm{wt}(v) \neq v_1$. Moreover, $\mathrm{wt}(u) \in \mathrm{col}(M_4)$ while $v_2, v_5 \notin \mathrm{col}(M_4)$. Thus $\mathrm{wt}(u) = v_6$. Also $\mathrm{wt}(v) \in \mathrm{col}(M_3)$ and $v_5, v_6 \notin \mathrm{col}(M_3)$, so $\mathrm{wt}(v) = v_2$. Since $\mathrm{wt}(w), \mathrm{wt}(x) \in \mathrm{col}(M_i)$ for all i, $\mathrm{wt}(w) = \mathrm{wt}(x) = v_1$.

Since $|R| = 2$, either $\mathcal{I}_{uv} \setminus \mathcal{I}_S = \{M_1\}$ and $\mathcal{I}_{wx} \setminus \mathcal{I}_S = \{M_2\}$, or $\mathcal{I}_{uv} \setminus \mathcal{I}_S = \{M_2\}$ and $\mathcal{I}_{wx} \setminus \mathcal{I}_S = \{M_1\}$.

Suppose that $\mathcal{I}_{uv} \setminus \mathcal{I}_S = \{M_1\}$ and $\mathcal{I}_{wx} \setminus \mathcal{I}_S = \{M_2\}$. Since $M_2 \in \mathcal{I}_{wx}$, $\mathrm{wt}(wx) = 0$, which implies that $M_3 \in \mathcal{I}_{wx}$. Either $M_2 \in \mathcal{I}_{vw}$ or $M_2 \notin \mathcal{I}_{vw}$.

I. $M_2 \in \mathcal{I}_{vw}$. Let $R' = \{uv, vw\}$.

II. $M_2 \notin \mathcal{I}_{vw}$. Then $\mathrm{wt}(vw) = 0$, so $M_1 \in \mathcal{I}_{vw}$. Let $R' = \{vw, wx\}$.

In either case, $\mathcal{I}_{R'} = \mathcal{MR}(H)$, so $G \notin \mathcal{F}_4(\text{full house})$ by Corollary 28. This is a contradiction.

Suppose that $\mathcal{I}_{uv} \setminus \mathcal{I}_S = \{M_2\}$ and $\mathcal{I}_{wx} \setminus \mathcal{I}_S = \{M_1\}$. Either $M_1 \in \mathcal{I}_{vw}$ or $M_1 \notin \mathcal{I}_{vw}$.

I. $M_1 \in \mathcal{I}_{vw}$. Let $R' = \{uv, vw\}$.

II. $M_1 \notin \mathcal{I}_{vw}$. Then $\mathrm{wt}(vw) = 1$, so $M_2 \in \mathcal{I}_{vw}$. Also, as can easily be checked, $M_3 \in \mathcal{I}_{vw}$. Let $R' = \{vw, wx\}$.

In either case, $\mathcal{I}_{R'} = \mathcal{MR}(H)$, so $G \notin \mathcal{F}_4(\text{full house})$ by Corollary 28. This is a contradiction.

Therefore $|S| \neq 4$, so $|G - H| \leq |S| \leq 3$ and $|G| \leq 8$. This contradicts the assumption that $|G| \geq 9$, so this case does not occur.

For every possible value of $\mathcal{I}_S$, we have reached a contradiction. Thus $|G-H| \geq 4$ is impossible, so $|G - H| \leq 3$ and $|G| \leq 8$. ☐

9 $H = P_4$

Lemma 47. *If $G \in \mathcal{F}_4(P_4)$, then $|G| \leq 8$.*

Proof. Suppose that $G \in \mathcal{F}_4(P_4)$ and $|G| \geq 8$ (i.e., $|G - H| \geq 4$). Then $G - H$ has no vertices with zero weight by Corollary 32. Assume that (R, S, T) is an optimal triple for G and P_4. Let $\mathcal{MR}(P_4) = \{M_1, M_2, M_3, M_4, M_5\}$ and $\mathcal{C}(P_4) = \{C_1 = \{M_1, M_2\}, C_2 = \{M_3, M_4\}, C_3 = \{M_5\}\}$ be as in Proposition 17(g). By properties P1 and P3, $\mathcal{I}_S \in \{\emptyset, C_1, C_2, C_3, C_1 \cup C_2, C_1 \cup C_3, C_2 \cup C_3\}$, so we have the following cases.

Case 1: $\mathcal{I}_S = \emptyset$**.** By Observation 26, if $v \in S$, then

$$\mathrm{wt}(v) \in \bigcap_{i=1}^{5} \mathrm{col}(M_i) = \{\vec{0},\ v_1 = (1,0,0,1)^T\}.$$

The rank-preserving tables for $[v_1]$ with respect to M_1, M_2, M_3, M_4, and M_5 are, respectively,

$$P_1 = \begin{bmatrix}1\end{bmatrix}, \quad P_2 = \begin{bmatrix}0\end{bmatrix}, \quad P_3 = \begin{bmatrix}1\end{bmatrix}, \quad P_4 = \begin{bmatrix}0\end{bmatrix}, \quad \text{and } P_5 = \begin{bmatrix}1\end{bmatrix}.$$

Since $P_1 = P_3 = P_5$ and $P_2 = P_4$ are complementary, an edge uv in R has either $\mathcal{I}_{uv} = \{M_1, M_3, M_5\}$ or $\mathcal{I}_{uv} = \{M_2, M_4\}$. Thus $\mathcal{I}_R \in \{\{M_1, M_2, M_3, M_4, M_5\}, \{M_1, M_3, M_5\}, \{M_2, M_4\}\}$. By property P4, $\mathcal{I}_R \neq \{M_2, M_4\}$. Therefore we have the following cases.

Subcase 1.1: $\mathcal{I}_R = \{M_1, M_2, M_3, M_4, M_5\}$**.** Since $\mathcal{I}_R = \mathcal{MR}(P_4)$, Corollary 39 implies that $T = \emptyset$ and $|G - H| \leq 2|R|$. By Theorem 36(b), $|R| \leq 2$, so $|G - H| \leq 4$ and $|G| \leq 8$.

Subcase 1.2: $\mathcal{I}_R = \{M_1, M_3, M_5\}$**.** Since $\mathcal{I}_R \cup \mathcal{I}_T = \mathcal{MR}(H)$, properties P1 and P3 imply that $\mathcal{I}_T = \{M_1, M_2, M_3, M_4\}$. By Corollary 37, $|R| \leq 1$ and $|T| \leq 2$. By Theorem 34, $|G - H| \leq 4$, so $|G| \leq 8$.

Case 2: $\mathcal{I}_S = \{M_1, M_2\}$**.** By Observation 26, if $v \in S$, then

$$\mathrm{wt}(v) \in \mathrm{col}(M_3) \cap \mathrm{col}(M_4) \cap \mathrm{col}(M_5) = \{\vec{0},\ v_1 = (0,1,0,1)^T,\ v_2 = (1,0,0,1)^T,\ v_3 = (1,1,0,0)^T\}.$$

The rank-preserving tables for $[v_1, v_2, v_3]$ with respect to M_3, M_4, and M_5 are, respectively,

$$P_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad P_4 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad \text{and } P_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Since $P_3 = P_5$, an edge in R is rank-preserving for M_3 if and only if it is also rank-preserving for M_5. Thus $\mathcal{I}_R \in \{\{M_1, M_2, M_3, M_4, M_5\}, \{M_1, M_2, M_3, M_5\}, \{M_1, M_2, M_4\}\}$. By property P4, $\mathcal{I}_R \neq \{M_1, M_2, M_4\}$. Therefore we have the following cases.

Subcase 2.1: $\mathcal{I}_R = \{M_1, M_2, M_3, M_4, M_5\}$**.** Since $\mathcal{I}_R = \mathcal{MR}(P_4)$, Corollary 39 implies that $T = \emptyset$ and $|G - H| \leq 2|R|$. By Theorem 36(b), $|R| \leq 2$, so $|G - H| \leq 4$ and $|G| \leq 8$.

Subcase 2.2: $\mathcal{I}_R = \{M_1, M_2, M_3, M_5\}$**.** Since $\mathcal{I}_R \cup \mathcal{I}_T = \mathcal{MR}(H)$, properties P1 and P3 imply that $\mathcal{I}_T = \{M_1, M_2, M_3, M_4\}$ or $\mathcal{I}_T = \{M_3, M_4\}$. In each of these cases, $|R| \leq 1$ and $|T| \leq 1$ by Theorem 36(b) and Corollary 37, implying that $|G - H| \leq 3$ and $|G| \leq 7$. This contradicts the assumption that $|G| \geq 8$, so these cases do not occur.

Case 3: $\mathcal{I}_S = \{M_3, M_4\}$**.** By Observation 26, if $v \in S$, then

$$\mathrm{wt}(v) \in \mathrm{col}(M_1) \cap \mathrm{col}(M_2) \cap \mathrm{col}(M_5) = \{\vec{0},\ v_1 = (0,0,1,1)^T,\ v_2 = (1,0,0,1)^T,\ v_3 = (1,0,1,0)^T\}.$$

The rank-preserving tables for $[v_1, v_2, v_3]$ with respect to M_1, M_2, and M_5 are, respectively,

$$P_1 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad \text{and } P_5 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since $P_1 = P_5$, an edge in R is rank-preserving for M_1 if and only if it is also rank-preserving for M_5. Thus $\mathcal{I}_R \in \{\{M_1, M_2, M_3, M_4, M_5\}, \{M_1, M_3, M_4, M_5\}, \{M_2, M_3, M_4\}\}$. By property P4, $\mathcal{I}_R \neq \{M_2, M_3, M_4\}$. Therefore we have the following cases.

Subcase 3.1: $\mathcal{I}_R = \{M_1, M_2, M_3, M_4, M_5\}$**.** Since $\mathcal{I}_R = \mathcal{MR}(P_4)$, Corollary 39 implies that $T = \emptyset$ and $|G - H| \leq 2|R|$. By Theorem 36(b), $|R| \leq 2$, so $|G - H| \leq 4$ and $|G| \leq 8$.

Subcase 3.2: $\mathcal{I}_R = \{M_1, M_3, M_4, M_5\}$**.** Since $\mathcal{I}_R \cup \mathcal{I}_T = \mathcal{MR}(H)$, properties P1 and P3 imply that $\mathcal{I}_T = \{M_1, M_2, M_3, M_4\}$ or $\mathcal{I}_T = \{M_1, M_2\}$. In each of these cases, $|R| \leq 1$ and $|T| \leq 1$ by Theorem 36(b) and Corollary 37, implying that $|G - H| \leq 3$ and $|G| \leq 7$. This contradicts the assumption that $|G| \geq 8$, so these cases do not occur.

Case 4: $\mathcal{I}_S = \{M_5\}$**.** By Observation 26, if $v \in S$, then

$$\mathrm{wt}(v) \in \mathrm{col}(M_1) \cap \mathrm{col}(M_2) \cap \mathrm{col}(M_3) \cap \mathrm{col}(M_4) = \{\vec{0},\ v_1 = (0,1,1,1)^T,\ v_2 = (1,0,0,1)^T,\ v_3 = (1,1,1,0)^T\}.$$

The rank-preserving tables for $[v_1, v_2, v_3]$ with respect to M_1, M_2, M_3, and M_4 are, respectively,

$$P_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad P_3 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad \text{and } P_4 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Since $P_2 = P_4$, an edge in R is rank-preserving for M_2 if and only if it is also rank-preserving for M_4. Thus

$$\mathcal{I}_R \in \{\{M_1, M_2, M_3, M_4, M_5\}, \{M_1, M_2, M_4, M_5\}, \{M_2, M_3, M_4, M_5\}, \{M_2, M_4, M_5\}, \{M_1, M_3, M_5\}, \{M_1, M_5\}, \{M_3, M_5\}\}.$$

By property P4, $\mathcal{I}_R \notin \{\{M_2, M_4, M_5\}, \{M_1, M_3, M_5\}, \{M_1, M_5\}, \{M_3, M_5\}\}$. Therefore we have the following cases.

Subcase 4.1: $\mathcal{I}_R = \{M_1, M_2, M_3, M_4, M_5\}$**.** Since $\mathcal{I}_R = \mathcal{MR}(P_4)$, Corollary 39 implies that $T = \emptyset$ and $|G - H| \leq |S| \leq 2|R|$. By Theorem 36(b), $|R| \leq 3$. If $|R| \leq 2$, then $|G - H| \leq 4$ and $|G| \leq 8$.

Suppose that $|R| = 3$. Then Theorem 36(b) implies that R consists of three edges e_1, e_2, and e_3 such that $\mathcal{I}_{e_1} \setminus \mathcal{I}_S = \{M_1\}$, $\mathcal{I}_{e_2} \setminus \mathcal{I}_S = \{M_2, M_4\}$, and $\mathcal{I}_{e_3} \setminus \mathcal{I}_S = \{M_3\}$.

Since the first row and column of P_1 and P_2 are the same, if an edge $e \in R$ is incident to a vertex of weight v_1, then either $\{M_1, M_2\} \subseteq \mathcal{I}_e$ or $\{M_1, M_2\} \subseteq \mathcal{MR}(H) \setminus \mathcal{I}_e$. Therefore e_1 and e_2 are not incident to vertices with weight v_1. Since the third row and column of P_2 and P_3 are the same, if an edge $e \in R$ is incident to a vertex of weight v_3, then either $\{M_2, M_3\} \subseteq \mathcal{I}_e$ or $\{M_2, M_3\} \subseteq \mathcal{MR}(H) \setminus \mathcal{I}_e$. Therefore e_2 and e_3 are not incident to vertices with weight v_3. Since $P_1[2] = P_3[2]$, if both vertices incident to an edge $e \in R$ have weight v_2, then $\{M_1, M_3\} \subseteq \mathcal{I}_e$ or $\{M_1, M_3\} \subseteq \mathcal{MR}(H) \setminus \mathcal{I}_e$. Therefore e_1 and e_3 each are incident to at least one vertex that does not have weight v_2.

Therefore e_1 must be incident to vertices with weights v_2 and v_3 (implying that $\mathrm{wt}(e_1) = 0$ since $\mathcal{I}_{e_1} = \{M_1\}$) or incident to vertices with weights v_3 and v_3 (implying that $\mathrm{wt}(e_1) = 1$). Each vertex incident to e_2 must have weight v_2, which implies that $\mathrm{wt}(e_2) = 1$. The edge e_3 must be incident to vertices with weights v_1 and v_1 (implying that $\mathrm{wt}(e_3) = 1$) or incident to vertices with weights v_1 and v_2 (implying that $\mathrm{wt}(e_3) = 0$).

Therefore there are at least three vertices u, v, and w in S such that u is incident to e_3, v is incident to e_2, w is incident to e_1, $\mathrm{wt}(u) = v_1$, $\mathrm{wt}(v) = v_2$, and $\mathrm{wt}(w) = v_3$. Let $R' = \{uv, vw\}$. Note that since $v_1, v_3 \notin \mathrm{col}(M_5)$, $M_5 \in \mathcal{I}_{R'}$. Suppose that $|S| \geq 5$. Then the vertices in $R' \cup e$ for any edge $e \in R$ form a proper subset of S. We now have the following possibilities for $\mathcal{I}_{R'}$.

I. $M_1 \notin \mathcal{I}_{R'}$. Then $\text{wt}(vw) = 1$, which implies that $M_2, M_3, M_4 \in \mathcal{I}_{R'}$. Since $\mathcal{I}_{R' \cup e_1} = \mathcal{MR}(H)$, $G \notin \mathcal{F}_4(P_4)$ by Corollary 28, which is a contradiction.

II. $M_2, M_4 \notin \mathcal{I}_{R'}$. Then $\text{wt}(uv) = 0$, which implies that $M_3 \in \mathcal{I}_{R'}$. Also $\text{wt}(vw) = 0$, which implies that $M_1 \in \mathcal{I}_{R'}$. Since $\mathcal{I}_{R' \cup e_2} = \mathcal{MR}(H)$, $G \notin \mathcal{F}_4(P_4)$ by Corollary 28, which is a contradiction.

III. $M_3 \notin \mathcal{I}_{R'}$. Then $\text{wt}(uv) = 1$, which implies that $M_1, M_2, M_4 \in \mathcal{I}_{R'}$. Since $\mathcal{I}_{R' \cup e_3} = \mathcal{MR}(H)$, $G \notin \mathcal{F}_4(P_4)$ by Corollary 28, which is a contradiction.

IV. $\mathcal{I}_{R'} = \mathcal{MR}(P_4)$. Since the vertices in R' are a proper subset of the vertices in S, $G \notin \mathcal{F}_4(P_4)$ by Corollary 28, which is a contradiction.

Since each case leads to a contradiction, our assumption that $|S| \geq 5$ must be false. Therefore $|S| \leq 4$, so $|G - H| \leq 4$ and $|G| \leq 8$.

Subcase 4.2: $\mathcal{I}_R = \{M_1, M_2, M_4, M_5\}$. Since $\mathcal{I}_R \cup \mathcal{I}_T = \mathcal{MR}(H)$, properties P1 and P3 imply that $\mathcal{I}_T = \{M_3, M_4, M_5\}$ or $\mathcal{I}_T = \{M_3, M_4\}$. In each of these cases, $|T| \leq 1$ by Corollary 37. Since $M_2 \in \mathcal{I}_{uv}$ if and only if $M_4 \in \mathcal{I}_{uv}$ for each edge $uv \in R$, $|R| \leq 2$ by Theorem 36(b). If $|R| = 1$, then $|G - H| \leq 3$, a contradiction. If $|R| = 2$, then Theorem 36(b) implies that R consists of an edge uv such that $\mathcal{I}_{uv} \setminus \mathcal{I}_S = \{M_2, M_4\}$ and another edge wx such that $\mathcal{I}_{wx} \setminus \mathcal{I}_S = \{M_1\}$. Since the first row and column of P_1, P_2, and P_4 are identical, we see that if any vertex, say u, has weight v_1, then the edge in R incident to the vertex must have either $\mathcal{I}_{uv} \setminus \mathcal{I}_S = \mathcal{I}_R \setminus \mathcal{I}_S$ or $\mathcal{I}_{uv} \setminus \mathcal{I}_S = \emptyset$. Neither of these cases occur, so u, v, w, and x each must have weight v_2 or v_3. As in Subcase 1.1 in Section 6, since $P_1[2,3]$ and $P_2[2,3] = P_4[2,3]$ are complementary, $|R| = 2$, and $|\bar{\mathcal{I}}_S| = 1$, we can apply Corollary 41 to conclude that $|S| = 3$, implying that $|G - H| \leq 4$ and $|G| \leq 8$.

Subcase 4.3: $\mathcal{I}_R = \{M_2, M_3, M_4, M_5\}$. Since $\mathcal{I}_R \cup \mathcal{I}_T = \mathcal{MR}(H)$, properties P1 and P3 imply that $\mathcal{I}_T = \{M_1, M_2\}$ or $\mathcal{I}_T = \{M_1, M_2, M_5\}$. In each of these cases, $|T| \leq 1$ by Corollary 37. Since $M_2 \in \mathcal{I}_{uv}$ if and only if $M_4 \in \mathcal{I}_{uv}$ for each edge $uv \in R$, $|R| \leq 2$ by Theorem 36(b). If $|R| = 1$, then $|G - H| \leq 3$, a contradiction. If $|R| = 2$, then Theorem 36(b) implies that R consists of an edge uv such that $\mathcal{I}_{uv} \setminus \mathcal{I}_S = \{M_2, M_4\}$ and another edge wx such that $\mathcal{I}_{wx} \setminus \mathcal{I}_S = \{M_3\}$. Note that the third row and column of P_2, P_3, and P_4 are identical; as in the previous case, none of u, v, x, w can have weight v_3, so each must have weight v_1 or v_2. Since $P_3[1,2]$ and $P_2[1,2] = P_4[1,2]$ are complementary, $|R| = 2$, and $|\bar{\mathcal{I}}_S| = 1$, we can apply Corollary 41 to conclude that $|S| = 3$, implying that $|G - H| \leq 4$ and $|G| \leq 8$.

Case 5: $\mathcal{I}_S = \{M_1, M_2, M_3, M_4\}$, $\mathcal{I}_S = \{M_1, M_2, M_5\}$, or $\mathcal{I}_S = \{M_3, M_4, M_5\}$. In each of these cases, by property P4, $\mathcal{I}_R = \{M_1, M_2, M_3, M_4, M_5\}$, so $T = \emptyset$ and $|G - H| \leq 2|R|$ by Corollary 39. In each of these cases, $|R| \leq 2$ by Corollary 37, so $|G - H| \leq 4$ and $|G| \leq 8$. □

10 All graphs in $\mathcal{F}_4(\mathbb{F}_2)$

Combining Lemmas 42 through 47 with Corollary 14, we have:

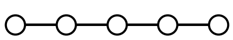
Theorem 48. *All graphs in $\mathcal{F}_4(\mathbb{F}_2)$ have 8 or fewer vertices.*

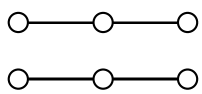
Theorem 3.1 in [DK06] implies that all graphs in $\mathcal{F}_4(\mathbb{F}_2)$ have 25 or fewer vertices. Because we have made a much more detailed analysis for the field $\mathbb{F}_2$, we have been able to greatly improve their bound in this single case. Since all graphs in $\mathcal{F}_4(\mathbb{F}_2)$ have 8 or fewer vertices, we can do an exhaustive search for all the graphs. In Appendix A, we list a few Magma functions sufficient to implement this search. These functions use the graph generation program "geng" distributed with Brendan McKay's Nauty program [McK90, Version 2.2]. This exhaustive search results in the 62 graphs displayed at the end of this section. Thus, recalling Observation 8, we have:

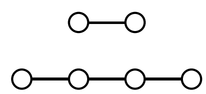
Theorem 49. *$\mathcal{F}_4(\mathbb{F}_2)$ consists of the 62 graphs listed at the end of this section. For any graph G,* $\mathrm{mr}(\mathbb{F}_2, G) \leq 3$ *if and only if no graph in $\mathcal{F}_4(\mathbb{F}_2)$ is induced in G.*

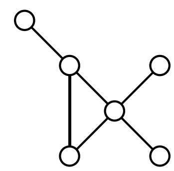
In the listing of the graphs in $\mathcal{F}_4(\mathbb{F}_2)$ that follows, the graphs are sorted by number of vertices. We have also tried to group similarly structured graphs together. Each graph is identified with a number and a graph6 code. The graph6 code is a compact representation of the adjacency matrix (and thus the zero/nonzero pattern of the matrices associated with the graph). The specification of the graph6 code is distributed with Nauty and can also be found on the Nauty website.

We now proceed with the listing of all 62 graphs in $\mathcal{F}_4(\mathbb{F}_2)$.


1 DDW


2 E?d_


3 E_GW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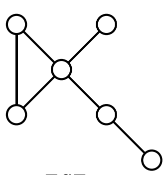
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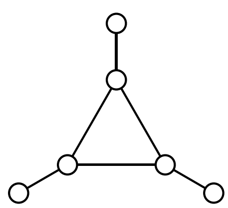
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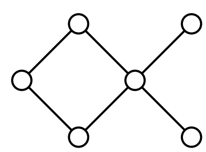
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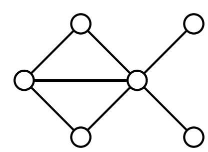
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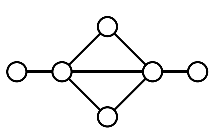
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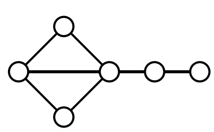
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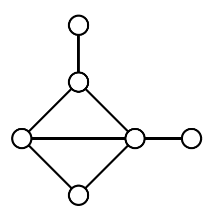
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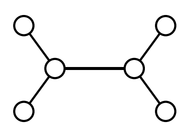
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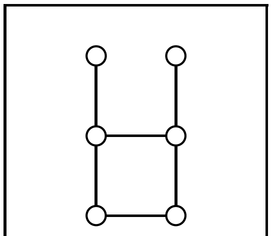
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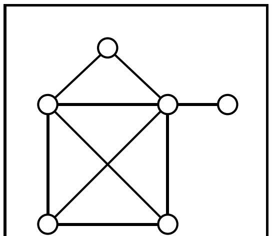
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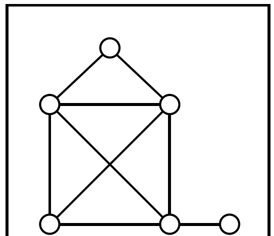
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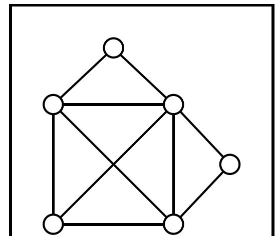
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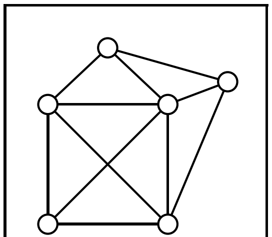
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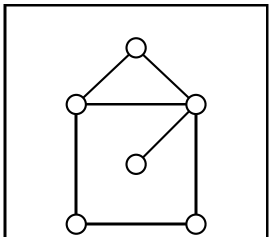
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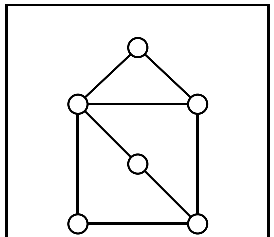
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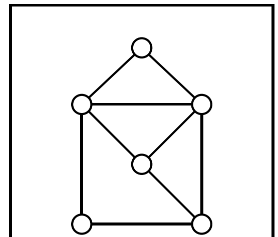
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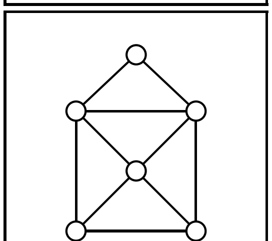
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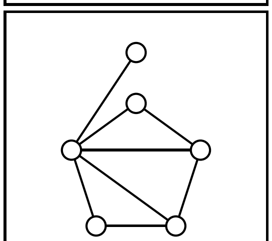
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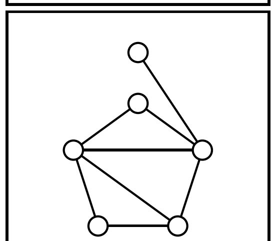
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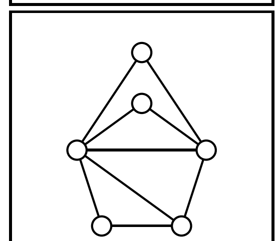
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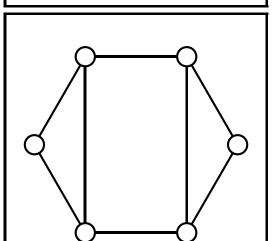
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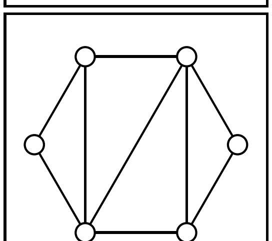
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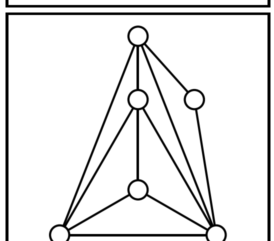
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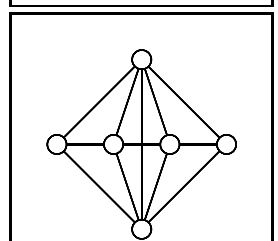
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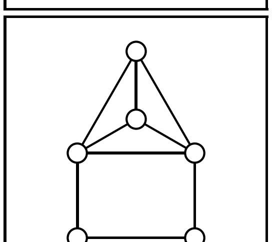
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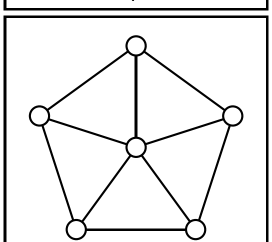
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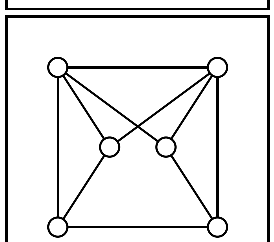
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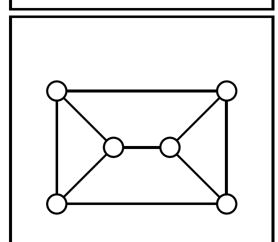
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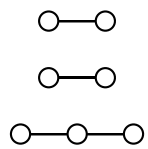
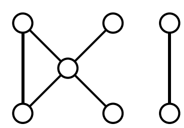

33 F@?M?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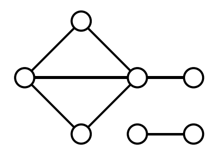
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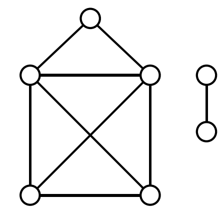
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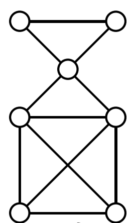
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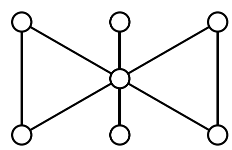
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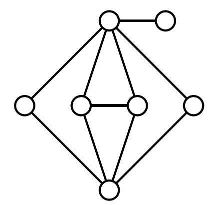
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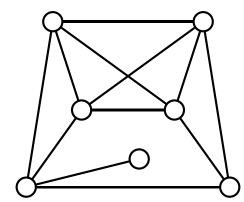
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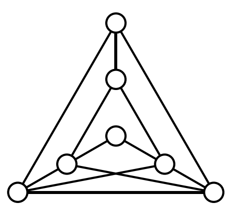
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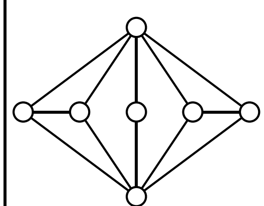
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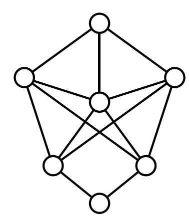
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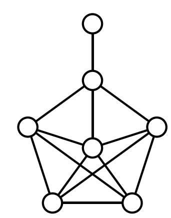
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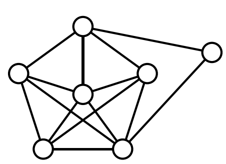
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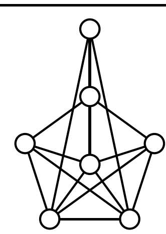
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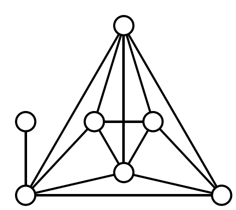
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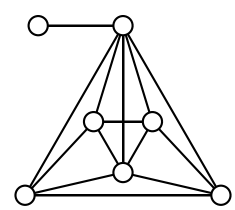
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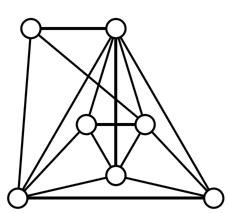
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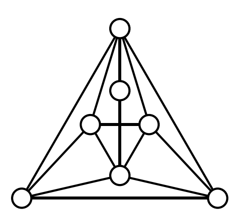
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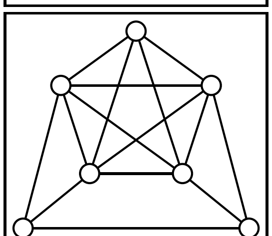
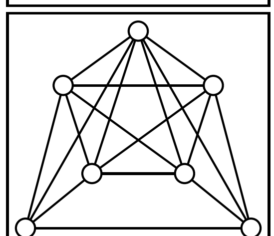
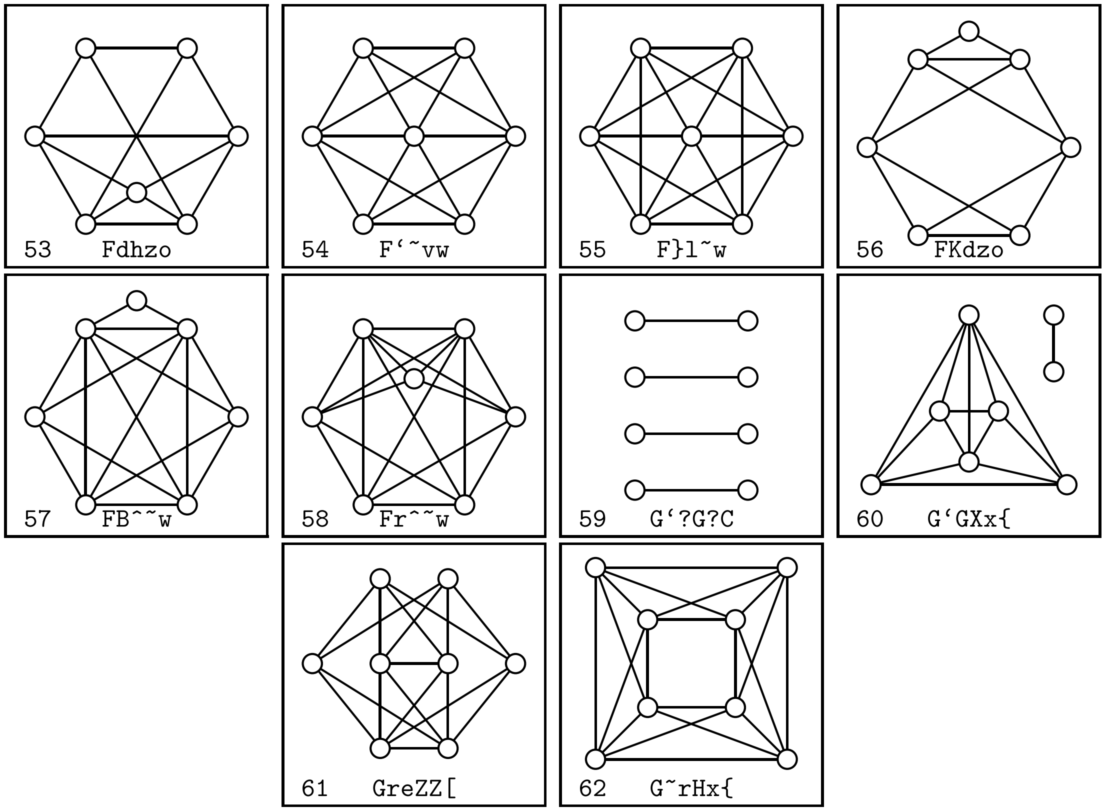

51 Fdlzw


52 Ft'~w

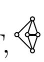


11 Graphs in $\mathcal{F}_4(F)$ for other fields

Many of the graphs in $\mathcal{F}_4(\mathbb{F}_2)$ are also in $\mathcal{F}_4(F)$ for any field F. This is the case with most of the disconnected graphs and the connected graphs with a cut vertex in the table.

We need the following elementary facts [BvdHL04].

Observation 50. For any field F

(a) $\mathrm{mr}(F, K_n) = 1$ for $n \geq 2$; $\mathrm{mr}(F, K_{2,3}) = \mathrm{mr}(F,$  $) = 2$; $\mathrm{mr}(F, \ltimes) = \mathrm{mr}(F, \mathrm{dart}) = 3$.

(b) $K_2 \in \mathcal{F}_1(F)$; $\ltimes, \mathrm{dart} \in \mathcal{F}_3(F)$.

(c) If $G = \cup_{i=1}^k G_i$, then $\mathrm{mr}(F, G) = \sum_{i=1}^k \mathrm{mr}(F, G_i)$.

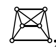
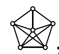
We will also need

Theorem 51 ([Fie69, BD05]). *Let F be any field and let G be a graph on n vertices. Then* $\mathrm{mr}(F, G) = n - 1$ *if and only if* $G = P_n$.

A stronger result was proved by Fiedler over $\mathbb{R}$ [Fie69] and his result was extended to any field, with some exceptions for $\mathbb{F}_3$, by Bento and Duarte [BD05].

Corollary 52. *For any field F,* $\mathrm{mr}(F, P_n) = n - 1$ *and* $P_n \in \mathcal{F}_{n-1}(F)$.

We will also utilize the following

Proposition 53. *Let* $\mathcal{E} = \{\text{full house}, G_1 =$ , $G_2 =$ , $P_3 \vee P_3\}$ *(G_1 is graph 40 minus the pendant vertex and G_2 is graph 44 minus the pendant vertex). Then for each $G \in \mathcal{E}$,* $\mathrm{mr}(\mathbb{F}_2, G) = 3$ *and* $\mathrm{mr}(F, G) = 2$ *for any $F \neq \mathbb{F}_2$. Moreover,* full house, $P_3 \vee P_3 \in \mathcal{F}_3(\mathbb{F}_2)$.

Proof. We already verified the first claim for the full house in the introduction. Taking complements of the others we find that $G_1^c = 2P_3$, $G_2^c = P_3 \cup K_2 \cup K_1$, and $(P_3 \vee P_3)^c = 2K_2 \cup 2K_1$. By Theorems 6 and 7 in [BvdHL04] and Theorems 11 and 15 in [BvdHL05], $\mathrm{mr}(F, G_1) = \mathrm{mr}(F, G_2) = \mathrm{mr}(F, P_3 \vee P_3) = 2$ for $F \neq \mathbb{F}_2$, while $\mathrm{mr}(\mathbb{F}_2, G_1) = \mathrm{mr}(\mathbb{F}_2, G_2) = \mathrm{mr}(\mathbb{F}_2, P_3 \vee P_3) = 3$. The final claim follows from Theorem 10. □

11.1 Disconnected graphs

Proposition 54. *If F is any field and $S_i \in \mathcal{F}_{\mathrm{mr}(S_i)}(F)$, $i = 1, \ldots, m$, then*

$$\bigcup_{i=1}^{m} S_i \in \mathcal{F}_{\mathrm{mr}(S_1)+\cdots+\mathrm{mr}(S_m)}(F).$$

Proof. This follows immediately from Observation 50(c) and the definition of $\mathcal{F}_{k+1}(F)$. □

Applying Observation 50(b), Corollary 52, and Proposition 54 to the disconnected graphs 2, 3, 33, 34, 35, and 59 in Section 10, we have

Theorem 55. *For any field F,*

$$\{2P_3, P_4 \cup K_2, P_3 \cup 2K_2, \ltimes \cup K_2, \text{dart} \cup K_2, 4K_2\} \subseteq \mathcal{F}_4(F).$$

Graphs 36 and 60 are full house $\cup K_2$ and $(P_3 \vee P_3) \cup K_2$. Since full house, $P_3 \vee P_3 \in \mathcal{F}_3(F)$ if and only if $F = \mathbb{F}_2$, graphs 36 and 60 are not in $\mathcal{F}_4(F)$ for any $F \neq \mathbb{F}_2$.

11.2 Connected graphs with a cut vertex

First, note that graph 1 in Section 10, P_5, is in $\mathcal{F}_4(F)$ for any field F by Corollary 52.

We now recall a definition and a known result.

Definition 56. Let G and H be graphs on at least two vertices, each having a vertex labeled v. Then $G \oplus_v H$ is the graph obtained from $G \cup H$ by identifying the two vertices labeled v. Similarly, if $G_1, \ldots, G_k$, $k \geq 2$, are graphs on at least two vertices, each with a vertex labeled v, let $G = G_1 \oplus_v G_2 \oplus_v \cdots \oplus_v G_k$ be the graph obtained by identifying the vertices labeled v in each of the graphs. We call G the *vertex sum* of the graphs $G_1, \ldots, G_k$ at v. Note that v is necessarily a cut vertex of a graph constructed in this way and that any graph with a cut vertex can be expressed as such a sum with $k \geq 2$.

The following theorem was proved over the real field in [Hsi01] and [BFH04]. In Appendix B, we give an easy proof of part (a) that holds for any field; part (b) then follows by induction. This same proof is also a key part of the proof of a more general result on the inertia set of a graph with a cut vertex (see Theorem 4.2 in [BHL]).

Theorem 57 ([Hsi01, BFH04]). *Let F be any field.*

(a) If G and H are graphs on at least two vertices, each having a vertex labeled v, then

$$\mathrm{mr}(F, G \bigoplus_v H) = \min\{\mathrm{mr}(F, G) + \mathrm{mr}(F, H),\ \mathrm{mr}(F, G - v) + \mathrm{mr}(F, H - v) + 2\}.$$

(b) Let $G_1, \ldots, G_k$, $k \geq 2$, be graphs on at least two vertices, each with a vertex labeled v. Then

$$\mathrm{mr}(F, G_1 \bigoplus_v G_2 \bigoplus_v \cdots \bigoplus_v G_k) = \min\{\sum_{i=1}^{k} \mathrm{mr}(F, G_i), \sum_{i=1}^{k} \mathrm{mr}(F, G_i - v) + 2\}.$$

This result reduces the calculation of the minimum rank of any graph with a cut vertex to a calculation for smaller graphs.

Corollary 58. $\mathrm{mr}(F, G \oplus_v H) \leq \mathrm{mr}(F, G) + \mathrm{mr}(F, H)$.

We can now establish one criterion for membership in $\mathcal{F}_4(F)$ for any field.

Theorem 59. *Let F be any field and let G be a graph satisfying all of the following*

(a) $|G| = 6$,

(b) $\mathrm{mr}(F, G) = 4$, *and*

(c) P_5 *is not induced in* G.

Then $G \in \mathcal{F}_4(F)$.

Proof. For each vertex v of G, $G - v$ is a graph on 5 vertices distinct from P_5. By Theorem 51, $\mathrm{mr}(F, G) < 5 - 1 = 4$. By Definition 7, $G \in \mathcal{F}_4(F)$. ☐

Proposition 60. *Graphs 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 18, 22, and 23 are all in $\mathcal{F}_4(F)$ for any field F.*

Proof. Each of these graphs has 6 vertices and P_5 is induced in none of them. Moreover, each graph is of the form $G \oplus_v K_2$, where $G \neq$ full house is a graph on 5 vertices. Let $G \oplus_v K_2$ be any of these graphs. By Theorem 57, Proposition 1, and Theorem 49,

$$\begin{aligned}\mathrm{mr}(F, G \bigoplus_v K_2) &= \min\{\mathrm{mr}(F, G) + \mathrm{mr}(F, K_2), \mathrm{mr}(F, G - v) + \mathrm{mr}(F, K_2 - v) + 2\} \\ &= \min\{\mathrm{mr}(\mathbb{F}_2, G) + \mathrm{mr}(\mathbb{F}_2, K_2),\ \mathrm{mr}(\mathbb{F}_2, G - v) + \mathrm{mr}(\mathbb{F}_2, K_2 - v) + 2\} \\ &= \mathrm{mr}(\mathbb{F}_2, G \bigoplus_v K_2) = 4.\end{aligned}$$

By Theorem 59, $G \oplus_v K_2 \in \mathcal{F}_4(F)$. ☐

We note that graphs 14 and 15, which contain the full house, have minimum rank 3 over any field $F \neq \mathbb{F}_2$, so are not in $\mathcal{F}_4(F)$ for $F \neq \mathbb{F}_2$.

We now consider in turn graphs 38 and 39.

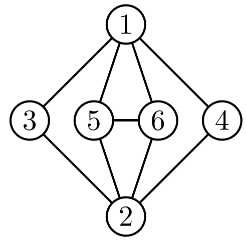


Figure 4: H = graph 39 minus the pendant vertex.

Graph 38 ( ): Applying Theorem 57(b) with $k = 4$, we have

$$\begin{aligned}\operatorname{mr}(F, \text{graph 38}) &= \min\{2\operatorname{mr}(F, K_3) + 2\operatorname{mr}(F, K_2),\ 2\operatorname{mr}(F, K_2) + 2\operatorname{mr}(F, K_1) + 2\} \\ &= \min\{2 + 2, 2 + 0 + 2\} = 4\end{aligned}$$

Theorem 57 also implies that $\operatorname{mr}(F, \ ) = 3$. By definition, graph 38 $\in \mathcal{F}_4(F)$.

Graph 39 ( ): Let F be any field. Let H be the graph obtained by deleting the pendant vertex in graph 39, labeled as in Figure 4. Since $\ltimes$ is induced in H, $\operatorname{mr}(F, H) \geq \operatorname{mr}(F, \ltimes) = 3$ by Observation 50. Moreover,

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix} \in \mathcal{S}(F, H)$$

and rank $A = 3$. Therefore $\operatorname{mr}(F, H) = 3$. By Theorem 57,

$$\begin{aligned}\operatorname{mr}(F, \text{graph } 39) &= \min\{\operatorname{mr}(F, H) + \operatorname{mr}(F, K_2),\ \operatorname{mr}(F, \ltimes) + \operatorname{mr}(F, K_1) + 2\} \\ &= \min\{3 + 1,\ 3 + 0 + 2\} = 4.\end{aligned}$$

Any graph obtained by deleting a vertex from graph 39 is one of H, $\ltimes \cup K_1$, , , or . By Observation 50, $\operatorname{mr}(F, \ltimes \cup K_1) = 3$. We just saw that has minimum rank 3. Since $K_{2,3}$ and have minimum rank 2 over any field by Observation 50, the graphs and each have minimum rank at most 3 by Corollary 58. By definition, graph 39 $\in \mathcal{F}_4(F)$ for every field F.

Summarizing,

Proposition 61. *Graphs 38 and 39 are in $\mathcal{F}_4(F)$ for any field F.*

The four remaining connected graphs with cut vertices, graphs 40, 44, 47, and 48, in the table do not belong to $\mathcal{F}_4(F)$ for $F \neq \mathbb{F}_2$. Let G be any of these graphs. Deleting the pendant vertex in G yields one of the last three graphs in Proposition 53, so by that result and Corollary 58, $\operatorname{mr}(F, G) \leq 2 + 1 = 3$ for $F \neq \mathbb{F}_2$.

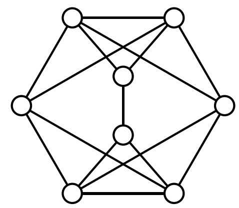


Figure 5: An 8 vertex graph in $\mathcal{F}_4(\mathbb{F}_2, P_4) \setminus \mathcal{F}_4(\mathbb{F}_2)$.

11.3 Summary

We have seen that 6 of the 8 disconnected graphs in Section 10 are in $\mathcal{F}_4(F)$ for all fields F, while 16 of 22 of the connected graphs with a cut vertex are in $\mathcal{F}_4(F)$ for all F.

We stated in the introduction that even if one is only interested in the minimum rank problem over $\mathbb{R}$, results obtained over $\mathbb{F}_2$ yield important insights. We have just observed that of the 30 graphs with vertex connectivity at most one in the list of 62 graphs in $\mathcal{F}_4(\mathbb{F}_2)$, 22 of these are also in $\mathcal{F}_4(F)$ for *any* field. While the discrepancy is significant, it is also the case that the amount of overlap is surprising. The analysis of the 2-connected graphs in Section 10 seems to be much more complicated with our present tools.

We have not found all graphs with vertex connectivity less than 2 in $\mathcal{F}_4(F)$, $F \neq \mathbb{F}_2$, by the above methods. For example, let F be any field with $\operatorname{char} F \neq 2$. Then $\operatorname{mr}(F, K_{3,3,3}) = 3$ and $\operatorname{mr}(F, K_{3,3,2}) = 2$ [BvdHL04]. Let $G = K_{3,3,3} \oplus_v K_2$. By Theorem 57,

$$\begin{aligned}\operatorname{mr}(F, G) &= \min\{\operatorname{mr}(F, K_{3,3,3}) + \operatorname{mr}(F, K_2),\ \operatorname{mr}(F, K_{3,3,2}) + \operatorname{mr}(F, K_1) + 2\} \\ &= \min\{3 + 1,\ 2 + 0 + 2\} = 4.\end{aligned}$$

But since for either of the two nonisomorphic graphs $K_{3,3,2} \oplus_v K_2$ arising from different choices of v, we have $\operatorname{mr}(F, K_{3,3,2} \oplus_v K_2) \leq \operatorname{mr}(F, K_{3,3,2}) + \operatorname{mr}(F, K_2) = 2 + 1 = 3$ by Corollary 58, it follows that $K_{3,3,3} \oplus_v K_2 \in \mathcal{F}_4(F)$. This graph did not occur in the table $\mathcal{F}_4(\mathbb{F}_2)$ because $\operatorname{mr}(\mathbb{F}_2, K_{3,3,3}) = 2$ [BvdHL04]. It is also easy to see that $K_{3,3,3} \cup K_2 \in \mathcal{F}_4(F)$ if $\operatorname{char} F \neq 2$. However, it is difficult to analyze the structure of all graphs in $\mathcal{F}_4(F)$ with vertex connectivity less than 2. It is difficult to understand even the graphs in $\mathcal{F}_4(F)$ that are of the form $G \oplus_v K_2$. Sometimes $G \in \mathcal{F}_3(F)$, but frequently it is not. We do know, however, that $\mathcal{F}_4(F)$ is infinite if F is $\mathbb{R}$ or $\mathbb{C}$ [Hal].

In examining the list of graphs in Section 10, we see that some of the bounds obtained in Sections 4–9 do not appear to be sharp. For instance, there is no graph in Section 10 with 8 vertices that has an induced P_4, even though the bound in Lemma 47 is 8 vertices. This is because there are graphs in $\mathcal{F}_4(\mathbb{F}_2, P_4)$ that are not in $\mathcal{F}_4(\mathbb{F}_2)$. For example, Figure 5 shows a graph on 8 vertices which is in $\mathcal{F}_4(\mathbb{F}_2, P_4)$ (when the induced P_4 contains both center vertices), as can be checked by hand or by using the Magma functions in the appendix. However, the graph is not in $\mathcal{F}_4(\mathbb{F}_2)$, since deleting one of the center vertices yields graph 56 in Section 10. This shows that Lemma 47 does indeed provide a sharp bound for the number of vertices in a graph in $\mathcal{F}_4(\mathbb{F}_2, P_4)$.

We have succeeded in obtaining a sharp bound on the number of vertices in a minimal forbidden subgraph for the class of graphs whose minimum rank is at most 3 over $\mathbb{F}_2$. We have also generated a complete list of these minimal forbidden subgraphs, thereby giving a structural characterization for the graphs having minimum rank 4 or more over $\mathbb{F}_2$. Since this result leads to a method for generating or recognizing all such graphs, it also leads to a theoretical procedure for determining whether a given graph has minimum rank at most 3 over $\mathbb{F}_2$.

A Magma programs

```
// We are working in F_2.
F:=FiniteField(2);

// This function returns all matrices in S(F_2,G) by adding
// all possible diagonal matrices to the adjacency matrix of G.
matrices_in_S:=function(graph)
    return {DiagonalMatrix(F,x)+AdjacencyMatrix(graph):
            x in Subsequences({x: x in F}, #Vertices(graph))};
end function;

// This function returns the minimum rank of a matrix by brute
// force computation.
minrank:=function(graph)
    return Min({Rank(m): m in matrices_in_S(graph)});
end function;

// This function returns the matrices in S(F_2,G) that attain
// the minimum rank.
minrank_matrices:=function(graph)
    return {m: m in matrices_in_S(graph) | Rank(m) eq minrank(graph)};
end function;

// This function returns true if and only if a subgraph of graph is
// isomorphic to a graph in graphlist
// (i.e., if graph is forbidden by graphlist).
isomorphic_subgraph:=function(graph,graphlist)
    if exists(t){<subgraph,fgraph>:
            subgraph in {sub<graph|s>: s in Subsets(Set(VertexSet(graph)))},
            fgraph in graphlist
            | IsIsomorphic(subgraph,fgraph)} then
         return true;
    else
        return false;
    end if;
end function;

// This is another version of the isomorphic_subgraph function.
isomorphic_subgraph:=function(graph,graphlist)
    for subgraph in {sub<graph|s>: s in Subsets(Set(VertexSet(graph)))} do
        if exists(t){ fgraph: fgraph in graphlist |
```

```
                    IsIsomorphic(subgraph,fgraph)} then
              return true;
        end if;
    end for;
    return false;
end function;

// This function appends a list of forbidden subgraphs with
// numvertices vertices to forbiddengraphs.  The geng program
// must be in the current directory.
generate_forbidden_graphs:=function(numvertices,forbiddengraphs)
    allgraphs:=OpenGraphFile("cmd geng "
        *IntegerToString(numvertices), 0, 0);
    while true do
        more, graph:=NextGraph(allgraphs);
        if more then
            if minrank(graph) ge 4
            and not isomorphic_subgraph(graph,forbiddengraphs) then
                Include(~forbiddengraphs,graph);
            end if;
        else
            break;
        end if;
    end while;
    return forbiddengraphs;
end function;

// Initialize the forbiddengraphs set and generate the forbidden
// subgraphs with 8 or fewer vertices.
forbiddengraphs:={};
for i in [1..8] do
    forbiddengraphs:=generate_forbidden_graphs(i,forbiddengraphs);
end for;

// Now forbiddengraphs contains all graphs in \mathcal{F}_4(F_2) as
// Magma graphs.
```

B Field independent proof of Theorem 57

First recall a definition, a well-known fact, and the statement of the theorem.

Definition. Let G and H be graphs on at least two vertices, each having a vertex labeled v. Then $G \oplus_v H$ is the graph obtained from $G \cup H$ by identifying the two vertices labeled v.

Lemma ([Nyl96])**.** *If F is any field and G is a graph with a vertex v, then* $\mathrm{mr}(F, G - v) \le \mathrm{mr}(F, G) \le \mathrm{mr}(F, G - v) + 2$.

Theorem ([Hsi01, BFH04]). *Let F be any field and let G and H be graphs on at least two vertices, each having a vertex labeled v. Then*

$$\operatorname{mr}(F, G \underset{v}{\oplus} H) = \min\{\operatorname{mr}(F,G) + \operatorname{mr}(F,H),\ \operatorname{mr}(F,G-v) + \operatorname{mr}(F,H-v) + 2\}. \tag{1}$$

Proof. Since v is a cut vertex of the connected graph $G \oplus_v H$, $(G \oplus_v H) - v = (G - v) \cup (H - v)$. By the lemma and Observation 50,

$$\operatorname{mr}(F, G \underset{v}{\oplus} H) \le \operatorname{mr}(F, G-v) + \operatorname{mr}(F, H-v) + 2.$$

Let v be the last vertex of G and the first vertex of H. Let

$$M = \begin{bmatrix} A & b \\ b^T & c_1 \end{bmatrix} \in S(F,G) \quad \text{and} \quad N = \begin{bmatrix} c_2 & d^T \\ d & E \end{bmatrix} \in S(F,H),$$

such that rank $M = \operatorname{mr}(F,G)$ and rank $N = \operatorname{mr}(F,H)$. Let

$$\hat{M} = \begin{bmatrix} A & b & 0 \\ b^T & c_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \hat{N} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & c_2 & d^T \\ 0 & d & E \end{bmatrix}.$$

Then $\hat{M} + \hat{N} \in S(F, G \oplus_v H)$ so

$$\begin{aligned} \operatorname{mr}(F, G \underset{v}{\oplus} H) &\le \operatorname{rank}(\hat{M} + \hat{N}) \\ &\le \operatorname{rank} \hat{M} + \operatorname{rank} \hat{N} = \operatorname{rank} M + \operatorname{rank} N \\ &= \operatorname{mr}(F,G) + \operatorname{mr}(F,H). \end{aligned}$$

This proves the $\le$ in (1).

Now let $M \in S(F, G \oplus_v H)$ with rank $M = \operatorname{mr}(F, G \oplus_v H)$. Write

$$M = \begin{bmatrix} A & b & 0 \\ b^T & c & d^T \\ 0 & d & E \end{bmatrix}.$$

Now

$$\begin{aligned} \operatorname{rank} A + \operatorname{rank} E &\le \operatorname{rank} \begin{bmatrix} A & b & 0 \\ 0 & d & E \end{bmatrix} && (2) \\ &\le \operatorname{rank} M && (3) \\ &\le \operatorname{rank} A + \operatorname{rank} E + 2. && (4) \end{aligned}$$

It follows that one of the three inequalities (2), (3), or (4) is an equality.

I. Suppose that (2) and (4) are strict inequalities. Then

$$\operatorname{rank} M = \operatorname{rank} \begin{bmatrix} A & b & 0 \\ 0 & d & E \end{bmatrix} = \operatorname{rank} A + \operatorname{rank} E + 1.$$

Consequently $\begin{bmatrix} b \\ d \end{bmatrix} \notin \text{col} \begin{bmatrix} A & 0 \\ 0 & E \end{bmatrix}$, so either $b \notin \text{col}(A)$ or $d \notin \text{col}(E)$. Assume $b \notin \text{col}(A)$. Then $b^T \notin \text{row}(A)$, so

$$\text{rank } M = \text{rank} \begin{bmatrix} A & b & 0 \\ b^T & c & d^T \\ 0 & d & E \end{bmatrix} > \text{rank} \begin{bmatrix} A & b & 0 \\ 0 & d & E \end{bmatrix},$$

a contradiction. Therefore, this case does not occur. So either (2) or (4) is an equality.

II. Suppose (2) is an equality. Then

$$\text{rank} \begin{bmatrix} A & 0 \\ 0 & E \end{bmatrix} = \text{rank} \begin{bmatrix} A & 0 & b \\ 0 & E & d \end{bmatrix}.$$

Thus $\begin{bmatrix} b \\ d \end{bmatrix} \in \text{col} \begin{bmatrix} A & 0 \\ 0 & E \end{bmatrix}$, which implies that $b = Au$, $d = Ev$ for some vectors u and v. Then

$$\hat{A} = \begin{bmatrix} A & Au \\ u^T A & u^T Au \end{bmatrix} = \begin{bmatrix} A & b \\ b^T & u^T Au \end{bmatrix} \in S(F, G)$$

and rank $\hat{A} =$ rank A. Similarly,

$$\hat{E} = \begin{bmatrix} v^T Ev & v^T E \\ Ev & E \end{bmatrix} = \begin{bmatrix} v^T Ev & d^T \\ d & E \end{bmatrix} \in S(F, H)$$

and rank $\hat{E} =$ rank E. It follows that

$$\begin{aligned} \text{mr}(F, G \underset{v}{\oplus} H) &= \text{rank } M \\ &\geq \text{rank} \begin{bmatrix} A & 0 \\ 0 & E \end{bmatrix} = \text{rank } A + \text{rank } E = \text{rank } \hat{A} + \text{rank } \hat{E} \\ &\geq \text{mr}(F, G) + \text{mr}(F, H). \end{aligned}$$

III. Suppose that (4) is an equality. Since $A \in S(F, G - v)$ and $B \in S(F, H - v)$, rank $A \geq \text{mr}(F, G - v)$ and rank $B \geq \text{mr}(F, H - v)$. Then

$$\text{mr}(F, G \underset{v}{\oplus} H) = \text{rank } M \geq \text{mr}(F, G - v) + \text{mr}(F, H - v) + 2.$$

Combining cases I, II, and III, we have proven the $\geq$ in (1). □

C Sage code to generate forbidden graphs

This appendix contains a translation of the code in Appendix A for Sage (see `http://www.sagemath.org`).

```
# This code is written for Sage 3.1.1
# See http://www.sagemath.org

def matrices_in_S(graph):
```

```
    """
    Return all matrices in S($\mathbb{F}_2$,graph) by adding all
    possible diagonal matrices to the adjacency matrix of the graph.
    The matrices are returned as matrices over $\mathbb{F}_2$.
    """
    F = FiniteField(2)
    A = graph.adjacency_matrix().change_ring(F)
    return [A+diagonal_matrix(list(diagonal))
            for diagonal in VectorSpace(F,graph.order())]


def minrank(graph):
    """
    Return the minimum rank of a graph over $\mathbb{F}_2$ by
    exhaustive enumeration.
    """
    return min(m.rank() for m in matrices_in_S(graph))

def minrank_matrices(graph):
    """
    Return the matrices in S($\mathbb{F}_2$,graph) that attain the
    minimum rank of G.
    """
    minimumrank = minrank(graph)
    return [m for m in matrices_in_S(graph) if m.rank() == minimumrank]

def isomorphic_subgraph(graph, graphlist):
    """
    Return True if a subgraph of graph is isomorphic to a graph in
    graphlist, otherwise return False.
    """
    # Get the numbers of vertices in graphlist
    n = graph.order()
    subgraph_sizes = set([g.order() for g in graphlist if g.order() <= n])
    for i in subgraph_sizes:
        graphlist_check = [g for g in graphlist if g.order() == i]
        for vertices in Combinations(graph.vertices(), i):
            subgraph = graph.subgraph(vertices)
            if any(subgraph.is_isomorphic(g) for g in graphlist_check):
                return True
    return False


def generate_forbidden_graphs(numvertices, forbiddengraphs):
    """
    Return all graphs having numvertices vertices and minimum rank at
```

```
    least 4 that do not contain a subgraph isomorphic to a graph in
    forbiddengraphs.
    """
    return [g for g in graphs(numvertices)
            if minrank(g) >= 4 and not isomorphic_subgraph(g, forbiddengraphs)]


# Wait for a while for these next few commands to complete.
forbiddengraphs=[]
for i in [1..8]:
    forbiddengraphs += generate_forbidden_graphs(i, forbiddengraphs)

# Now forbiddengraphs contains all graphs in F_4(F_2) as
# Sage graphs.
```

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