Linear Algebra Exam 1  
Math 80, Jason Grout, Spring 2014

Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, ask me for more paper. Circle all final answers. Be clear and complete in your reasoning and answers. If you’re asking yourself whether to include something in your answer, remember: when in doubt, write it out.

Justify all answers. No calculators.

Name: ______________________________________________________________________

1. Convert the following system of equations to the indicated forms. Do not solve the system of equations.

\[
\begin{align*}
3x + 2y - z &= 5 \\
4x - y &= 0 \\
x + 3y - 2z &= 3
\end{align*}
\]

(a) Matrix equation \( A\vec{x} = \vec{b} \)

\[
\begin{bmatrix}
3 & 2 & -1 \\
4 & -1 & 0 \\
1 & 3 & 2
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix} =
\begin{bmatrix}
5 \\
6 \\
3
\end{bmatrix}
\]

(b) Linear combination of vectors equals a vector

\[
\begin{bmatrix}
3 \\
4 \\
1
\end{bmatrix} x +
\begin{bmatrix}
2 \\
-1 \\
3
\end{bmatrix} y +
\begin{bmatrix}
-1 \\
0 \\
-2
\end{bmatrix} z =
\begin{bmatrix}
5 \\
6 \\
3
\end{bmatrix}
\]

(c) Augmented matrix

\[
\begin{bmatrix}
3 & 2 & -1 ; 5 \\
4 & -1 & 0 ; 0 \\
1 & 3 & 2 ; 3
\end{bmatrix}
\]
2. The RREF augmented matrices below represent systems of equations. For each matrix, if the system has a solution, give the solution vector \((x_1, x_2, x_3, x_4)\) as a linear combination of vectors (e.g., in parametric form, like many of the homework problems) and describe geometrically what the solution set looks like (for example, a line, a plane, whether or not it goes through the origin, etc.). If the system does not have a solution, write "inconsistent system", and justify your answer with a sentence or two.

(a) \[
\begin{bmatrix}
1 & 0 & -5 & 2 & 0 \\
0 & 1 & 2 & 4 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

Inconsistent. The last equation is \(0=1\), which can't be true.

(b) \[
\begin{bmatrix}
1 & 2 & 0 & 4 & 2 \\
0 & 0 & 1 & -2 & 1 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
\end{pmatrix} = \begin{pmatrix}
-2 \\
1 \\
0 \\
0 \\
\end{pmatrix} x_1 + \begin{pmatrix}
-4 \\
0 \\
2 \\
1 \\
\end{pmatrix} x_4 + \begin{pmatrix}
2 \\
0 \\
0 \\
0 \\
\end{pmatrix}
\]

This is a plane that does not go through the origin.

(c) \[
\begin{bmatrix}
1 & 0 & 0 & 0 & 3 \\
0 & 1 & 0 & 0 & 2 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 2 \\
\end{bmatrix}
\]

\[
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
\end{pmatrix} = \begin{pmatrix}
3 \\
2 \\
0 \\
2 \\
\end{pmatrix}
\]

This is a point (not at the origin).
3. Let $A$, $B$, $C$, and $D$ be invertible 5 by 5 matrices such that $ABC^{-1} = D$. Solve for $B$. Show each individual step of your computation.

\[
ABC^{-1} = D \\
A^{-1}ABC^{-1} = A^{-1}D \\
I = A^{-1}D \\
BC^{-1} = A^{-1}DC \\
B = A^{-1}DC
\]

4. True/False: Every nonzero 3 by 3 matrix has an inverse. \underline{False}

If True, justify your answer. If False, give an example of a nonzero 3 by 3 matrix that does not have an inverse and justify your answer.

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\] This matrix does not have $I$ as its ref, so it is not invertible.

(there are lots of other justifications for it not being invertible).
5. Show that for a general invertible matrix $A$, the inverse of $A^T$ is $(A^{-1})^T$. Justify each step by stating the general property of matrix arithmetic that you are using.

\[
A^T (A^{-1})^T = (A^T A)^T \quad \text{since } (AB)^T = B^T A^T
\]

\[
= I^T \quad \text{since } A^{-1}A = I \text{ by definition of inverse}
\]

\[
= I \quad \text{since } I \text{ has zeros off of the diagonal.}
\]

6. Find the matrix $2 \times 2$ matrix $A$ for the linear transformation $T(x) = Ax$ that does the following operations, in order: Rotates by 180 degrees, then flips over the $y$-axis, then stretches horizontally by a factor of 2. Make sure to justify your work.

Let's see what happens to $(0)$ and $(1)$.

\[
\begin{align*}
(0) & \xrightarrow{\text{rotate}} (-1) \xrightarrow{\text{flip}} (0) \xrightarrow{\text{stretch}} (3) \\
(1) & \xrightarrow{\text{rotate}} (0) \xrightarrow{\text{flip}} (-1) \xrightarrow{\text{stretch}} (-1)
\end{align*}
\]

Since \((ab) (1) = (a)\), and \((ab) (1) = (a)\), we have

\[
(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{and} \quad (1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \text{ so the matrix is}
\]

\[
\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}
\]
7. Let $A = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$.

(a) Find $A^{-1}$ using Gaussian elimination row reduction or show that the inverse does not exist.

$\begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_2} \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & -1 & 0 \end{bmatrix} \xrightarrow{R_1 \rightarrow -R_1 + R_2} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix} \xrightarrow{R_3 \rightarrow -R_3 + R_2} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$

So $A^{-1} = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 2 & -2 \\ 1 & -1 & 1 \end{bmatrix}$

(b) Check your work by multiplying $AA^{-1}$. Explain how it serves as a check of your work.

$\begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \\ -1 & 2 & -2 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1^{\text{st}} \text{col} \\ 2^{\text{nd}} \text{col} \\ 3^{\text{rd}} \text{col} \end{bmatrix}$

Since the matrices multiply to the identity, they are inverses.
8. Let $T(x, y, z) = (-2x + y - z, 2x, 2x - y + z)$ be a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$.

(a) Find a matrix $A$ so that $T(\vec{x}) = A\vec{x}$. (remember to circle your answer)

\[
\begin{bmatrix}
-2 & 1 & -1 \\
2 & 0 & 0 \\
2 & -1 & 1
\end{bmatrix}
\]

(b) What are the domain, codomain, and range of $T$?

**Domain**: $\mathbb{R}^3$

**Codomain**: $\mathbb{R}^3$

**Range**: $\text{span}\left\{ \left( \frac{-2}{2}, 1, 1 \right) \right\}$

(c) Is $T$ one-to-one? Explain.

No - the ref of $A$ has a free variable column.

(d) Is $T$ onto? Explain.

No - the ref of $A$ has a row of zeroes.
9. Let $S$ be the set of vectors $\{(3, 2, 1), (1, 3, 5), (3, 5, 2), (1, -2, 0)\}$

(a) Is $S$ linearly independent? Justify your answer. If $S$ is linearly dependent, find a nontrivial linear combination of the vectors that equals 0.

\[ \begin{pmatrix} 3 & 3 & 1 \\ 2 & 3 & 5 \\ 1 & 2 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \]

\[ -\frac{9}{7} \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + \frac{1}{7} \begin{pmatrix} 3 \\ 5 \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{3}{2} \\ \frac{5}{2} \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \]

(b) Does $S$ span $\mathbb{R}^3$? Explain your answer.

Yes. From the result above, we see that the transformation is onto, so the columns span the codomain, $\mathbb{R}^3$. 
10. List five things that are equivalent to a square matrix being invertible (i.e., if the matrix is invertible, these things are all true, and if the matrix is not invertible, these things are all false). Explain each of your answers. Be precise and complete in how you use the terminology.

(a)

(b)

(c) Various. See the invertible matrix theorem in section 2.3, for example.

(d)

(e)