1. Find the determinant of \( A \) using cofactor expansion along the second column:

\[
A = \begin{bmatrix}
-2 & 2 & -4 \\
0 & 1 & 3 \\
-3 & 4 & 1
\end{bmatrix}
\]

Answer:

\[
-2 \det \begin{pmatrix} 0 & 3 \\ -3 & 1 \end{pmatrix} + 1 \det \begin{pmatrix} -2 & -4 \\ 3 & 1 \end{pmatrix} - 4 \det \begin{pmatrix} -2 & -4 \\ 0 & 3 \end{pmatrix} = -2(9) + 1(-14) - 4(-6) = -8
\]
2. Let $B = \{(−1, 2), (4, −9)\}$ and $C = \{(−1, −2), (0, −1)\}$. Also, let $[\vec{y}]_B = (3, −1)$.

(a) What is $\vec{y}$ (in standard coordinates)?

Answer: Take the appropriate linear combination of $B$ basis vectors: $2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} 4 \\ -9 \end{bmatrix} = \begin{bmatrix} -7 \\ 15 \end{bmatrix}$, or you could compute the same linear combination using matrix-vector multiplication: $B[y]_B = \begin{bmatrix} -1 & 4 \\ 2 & -9 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -7 \\ 15 \end{bmatrix}$.

(b) Find a matrix $A$ that will convert from $B$ coordinates to $C$ coordinates (i.e., a matrix $A$ so that for a vector $\vec{x}$, $A[\vec{x}]_B = [\vec{x}]_C$).

Answer: As in the part above, multiplying by $B$ converts from $B$ coordinates to standard coordinates, multiplying by $C$ converts from $C$ coordinates to standard coordinates, and multiplying by $C^{-1}$ converts from standard coordinates to $C$ coordinates. In other words, $C^{-1}B[y]_B = C^{-1}[y]_E = [y]_C$. We can find $C^{-1}$ from RREF $C$. So the matrix is $C^{-1}B = \begin{bmatrix} -1 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} -1 & 4 \\ 2 & -9 \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ -4 & 17 \end{bmatrix}$.

For practice, draw the diagram illustrating the conversion, and label the first arrow from $B$ to $E$ with the matrix $B$, and the arrow from $E$ to $C$ with the matrix $C^{-1}$.

(c) What is $[\vec{y}]_C$, i.e., the coordinates of $\vec{y}$ relative to $C$?

Answer: We could use our answer from part (a), $C^{-1}[y]_E = \begin{bmatrix} -1 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} -7 \\ 15 \end{bmatrix} = \begin{bmatrix} 7 \\ -29 \end{bmatrix}$, or we could use our answer from part (b) with the original $[y]_B$: $\begin{bmatrix} 1 & -4 \\ -4 & 17 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 7 \\ -29 \end{bmatrix}$.
3. Let \( A = \begin{bmatrix} -4 & 5 & -35 & 2 & -6 \\ -2 & 1 & -13 & -3 & -8 \\ 5 & -5 & 40 & 1 & 12 \\ 3 & -5 & 30 & -6 & -2 \end{bmatrix} \), \( \text{rref}(A) = \begin{bmatrix} 1 & 0 & 5 & 0 & 0 \\ 0 & 1 & -3 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \).

(a) Find a basis for the row space of \( A \).
Answer: a basis for the row space is a list of row vectors that spans the row space. One such basis is given by the nonzero rows of the RREF: \( \{(1,0,5,0,0), (0,1,-3,0,-2), (0,0,0,1,2)\} \).

(b) Find a basis for the column space of \( A \).
Answer: a basis for the column space is a list of column vectors that spans the column space. One such basis is given by the pivot columns of the original matrix:
\[
\left\{ \begin{bmatrix} -4 \\ -2 \\ 5 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ -5 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 1 \\ -6 \end{bmatrix} \right\}
\]

(c) Find a basis for the null space of \( A \).
Answer: This is just the vectors that we get out of our normal parametric solution to \( A\vec{x} = 0 \). We use the RREF of \( A \) to get
\[
\begin{align*}
x_1 &= -5x_3 \\
x_2 &= 3x_3 + 2x_5 \\
x_4 &= -2x_5
\end{align*}
\]
so we get
\[
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} 0 \\ 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} x_5
\]
so a basis is given by \( \left\{ \begin{bmatrix} -5 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\} \).

(d) Find the rank and nullity of \( A \) (clearly label each).
Answer: the rank of a matrix is the dimension of the column space, which is 3. The nullity of a matrix is the dimension of the null space, which is 2.
4. Let $A$ be a square matrix. Let $V = \{ \vec{x} \text{ where } A\vec{x} = \vec{x} \}$ (in other words, $V$ is the set of all vectors $\vec{x}$ such that $A\vec{x} = \vec{x}$). Is $V$ a vector space? Explicitly check the conditions for a vector space.

Answer:

Check for closure under vector addition: Let $\vec{x}$ and $\vec{y}$ be any vectors in $V$. That means that $A\vec{x} = \vec{x}$ and $A\vec{y} = \vec{y}$ (because of the definition of $V$). We need to check to see if $\vec{x} + \vec{y}$ is also in $V$, or in other words, we need to check if $A(\vec{x} + \vec{y}) = \vec{x} + \vec{y}$. Here’s the work for the check: $A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y} = \vec{x} + \vec{y}$. The first equality comes from matrix multiplication being distributive, and the second equality comes from what we know about $\vec{x}$ and $\vec{y}$ from above. So yes, $V$ is closed under vector addition.

Check for closure under scalar multiplication: Let $\vec{x}$ be in $V$. That means that $A\vec{x} = \vec{x}$. Also, let $c$ be any real number. We need to check that $c\vec{x}$ is in $V$, or in other words, we need to check that $A(c\vec{x}) = c\vec{x}$. Let’s do it: $A(c\vec{x}) = cA\vec{x} = c\vec{x}$. The first equality comes from the fact that $Ac = cA$, the second comes from what we know about $\vec{x}$ from above. So yes, the set $V$ is closed under scalar multiplication.

We also note that $V$ is not empty, since it has at least the vector $\vec{0}$, since $A\vec{0} = \vec{0}$.

Since $V$ is closed under vector addition and scalar multiplication (and $V$ has at least one vector in it), $V$ is a vector space.
5. Let $V = \{(x, y) \text{ where } x \text{ and } y \text{ are real numbers and } y > 0\}$. Define addition on $V$ by $(a, b) \oplus (c, d) = (a + c - 3, bd)$. (Note that I'm writing vector addition as $\oplus$ to make it clear when we are using this new definition). Does $V$ have a zero vector (a vector $\vec{0}$ so that $\vec{v} \oplus \vec{0} = \vec{v}$ for every $\vec{v} \in V$)? If so, what is it? If not, why not? Justify your answer using only the definition of a zero vector.

Answer: We need a vector $\vec{0} = (c, d)$ so that every vector in $V$ added to $(c, d)$ results in the same vector from $V$. Let $(a, b)$ be any vector in $V$. Then we need $(a, b) \oplus (c, d) = (a, b)$. Let’s do the addition: $(a, b) \oplus (c, d) = (a + c - 3, bd)$. So we must have that $a = a + c - 3$ and $bd = b$. This means that $c = 3$ and $d = 1$. So our zero vector is $\vec{0} = (3, 1)$. 

6. In this problem, a “vector” is a 2 by 2 matrix. Let

\[ A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 4 \\ -1 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 5 & -6 \\ 3 & -10 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & -32 \\ 6 & -24 \end{bmatrix}, \quad E = \begin{bmatrix} -5 & 94 \\ -16 & 66 \end{bmatrix} \]

(a) Is the set \( \{A, B, C, D, E\} \) linearly independent? Why or why not?

Answer: The coordinate vectors for \( A, B, C, D, \) and \( E \) relative to the standard basis for 2 by 2 matrices are the columns of RREF \( E \). We see from the RREF in RREF \( E \) that the coordinate vectors are not linearly independent, so the set \( \{A, B, C, D, E\} \) is also not linearly independent.

(b) Give a basis for the vector space \( V = \text{span}\{A, B, C, D, E\} \). Call this basis \( B \). [Hint: Since “vectors” in \( V \) are 2 by 2 matrices, \( B \) should be a set of 2 by 2 matrices.]

A basis for the column thespace of the matrix in RREF \( E \) is the coordinate vectors for \( A, B, \) and \( D \), so a basis for the span of \( \{A, B, C, D, E\} \) is the corresponding matrices: \( \{A, B, D\} \). Note that the basis is a list of 2 by 2 matrices.
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(c) What is the dimension of the vector space \( V = \text{span}\{A, B, C, D, E\} \)? (as always, justify your answer)
There are 3 vectors in our basis \( B \), so the dimension is 3.

(d) Write and clearly label the coordinates \([A]_B\), \([B]_B\), \([C]_B\), \([D]_B\), and \([E]_B\) (i.e., the coordinates for each matrix relative to your basis \( B \) from above).
Answer: We can read these off from RREF E, since the RREF of a matrix tells how the columns depend on each other:

\[
[A]_B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\
[B]_B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\
[C]_B = \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix} \\
[D]_B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
[E]_B = \begin{bmatrix} 2 \\ -2 \\ -3 \end{bmatrix}
\]

You can check that, for example \( E = 2A - 2B - 3D \).
7. Suppose $A$ and $B$ are 2 by 2 matrices and $\det(A) = 2$ and $\det(B) = 5$. What is $\det(3A^T B^{-1})$? As always, show all justifying work for your answer.

Answer: Since $\det(A^T) = \det(A)$, and $\det(B^{-1}) = \frac{1}{\det(B)}$, and for a 2 by 2 matrix, $\det(cA) = c^2 \det(A)$, we have $\det(3A^T B^{-1}) = \frac{3^2 \det(A)}{\det(B)} = \frac{18}{5}$.

8. $A$ is a 20 by 15 matrix with rank 10. $T$ is the linear transformation associated with $A$ (i.e., $T(\vec{x}) = A\vec{x}$). Is $T$ one-to-one? Is $T$ onto? Justify your answer.

Answer: The nullity of $T$ is $15 - 10 = 5$, so the nullspace of $T$ is not just the zero vector, so $T$ is not one-to-one. Another way to see this is that there are free variables in $T\vec{x} = \vec{b}$ since there are only 10 pivot columns (since rank is 10), so there must be 5 free variables, so $T\vec{x} = \vec{b}$ must have infinite solutions.

Since the rank of $T$ is 10, that means that there are 10 rows of RREF of $T$ that have pivots, so there are also 10 rows of RREF of $T$ that don’t have pivots, and so are zero rows. Since RREF of $T$ has zero rows, $T$ is not onto.
Here are some possibly useful RREFs. If you use any of these for any reason, make sure to cite which one you are using.

A. \[
\begin{bmatrix}
-1 & 4 & 1 & 0 \\
2 & -9 & 0 & 1
\end{bmatrix}
\xrightarrow{rref}
\begin{bmatrix}
1 & 0 & -9 & -4 \\
0 & 1 & -2 & -1
\end{bmatrix}
\]

B. \[
\begin{bmatrix}
-2 & 2 & -4 \\
n & 1 & 3 \\
-3 & 4 & 1
\end{bmatrix}
\xrightarrow{rref}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

C. \[
\begin{bmatrix}
-1 & 0 & 1 & 0 \\
-2 & -1 & 0 & 1
\end{bmatrix}
\xrightarrow{rref}
\begin{bmatrix}
1 & 0 & -1 & 0 \\
0 & 1 & 2 & -1
\end{bmatrix}
\]

D. \[
\begin{bmatrix}
1 & 3 & 0 & 1 \\
-1 & 4 & -1 & 4 \\
5 & -6 & 3 & -10 \\
3 & -32 & 6 & -24 \\
-5 & 94 & -16 & 66
\end{bmatrix}
\xrightarrow{rref}
\begin{bmatrix}
1 & 0 & 0 & -8 \\
0 & 1 & 0 & 3 \\
0 & 0 & 1 & 16 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

E. \[
\begin{bmatrix}
1 & -1 & 5 & 3 & -5 \\
3 & 4 & -6 & -32 & 94 \\
0 & -1 & 3 & 6 & -16 \\
1 & 4 & -10 & -24 & 66
\end{bmatrix}
\xrightarrow{rref}
\begin{bmatrix}
1 & 0 & 2 & 0 & 2 \\
0 & 1 & -3 & 0 & -2 \\
0 & 0 & 0 & 1 & -3 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

F. \[
\begin{bmatrix}
1 & 3 \\
0 & 1
\end{bmatrix}
\xrightarrow{rref}
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]

G. \[
\begin{bmatrix}
-1 & 4 \\
-1 & 4
\end{bmatrix}
\xrightarrow{rref}
\begin{bmatrix}
1 & -4 \\
0 & 0
\end{bmatrix}
\]

H. \[
\begin{bmatrix}
5 & -6 \\
3 & -10
\end{bmatrix}
\xrightarrow{rref}
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]

I. \[
\begin{bmatrix}
3 & -32 \\
6 & -24
\end{bmatrix}
\xrightarrow{rref}
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]

J. \[
\begin{bmatrix}
-5 & 94 \\
-16 & 66
\end{bmatrix}
\xrightarrow{rref}
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]