Linear Algebra Final

Math 80, Jason Grout, Spring 2012 Name: ______________________

> Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page and clearly indicate that you have done so. Circle all your final answers. Be clear and complete in your reasoning and answers; when in doubt, write it out.
> After you have handed back the noncalculator portion, you may use a calculator. If you do a calculation using a calculator, write down the input and output and indicate that you computed the output on a calculator

Justify all answers

1 NO CALCULATOR

1. (4 pts) Let A, B, C, and D be invertible $n \times n$ matrices for which $A = C^{-1}D^{-1}B$. Solve this equation for D, so that D is the product of three separate matrices. Show each individual step of your computation.

2. (4 pts) Suppose that the *augmented* matrix for a system of linear equations has been reduced by row operations to the given reduced row-echelon form. Solve the system if it is consistent (give all solutions as a linear combination of vectors) or tell what is wrong if the system is not consistent.

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & -2 & 0 & 4 \\ 0 & 0 & 1 & 2 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} =$$

3. (4 pts) Let M be the matrix below. Is M invertible? Justify your answer by explicitly computing some quantity about M and give a short explanation why the quantity justifies your answer. You do not need to compute M^{-1} to answer this question.

$$M = \begin{bmatrix} 1 & 3 & -4 & 2 \\ 0 & 2 & 5 & 1 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

4. (4 pts) Set up a matrix equation $A \begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix} = \vec{b}$ so that the solution u_0, u_1, u_2 gives the parabola $y = u_0 + u_1 x + u_2 x^2$ that passes through the points $(1, 1)$, $(2, 3)$, $(3, -2)$. You do not need to solve this system. Your answer should be a matrix A and a vector $\vec{b}$.

5. (4 pts) Let A be an $n \times n$ matrix. Let S be the set of all vectors $\vec{v}$ in $\mathbb{R}^n$ such that $A\vec{v} = \vec{0}$. Either show that S is a vector subspace of $\mathbb{R}^n$ or give a reason why S is not a subspace. Justify your answer completely by checking the conditions for a vector space.

6. (4 pts) Find the eigenvalues and associated eigenvectors for the matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}$. Clearly label which eigenvalue goes with which eigenvector.

7. (2 pts each part) The reduced row echelon form of

$$A = \begin{bmatrix} 1 & 2 & -5 & 1 & 14 \\ -3 & -6 & 15 & -1 & -36 \\ 5 & 10 & -22 & 4 & 61 \\ 2 & 4 & -7 & 2 & 22 \end{bmatrix} \quad \text{is} \quad \begin{bmatrix} 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Find a basis for the row space of A. (Hint: this will be a set of vectors)

(b) Find a basis for the column space of A.

(c) Find the rank of A. As always, justify your answer.

2 CALCULATOR SECTION

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Hand back the noncalculator portion of the test before pulling out your calculator.

8. (4 pts) Is the set $\{x^2 + 2x + 1,\ -x^2 - 2,\ 2x^2 + x - 1,\ -x^2 + 3x + 1\}$ a linearly independent set in the vector space $P_2(x)$? Justify your answer. (Hint: you do not need to do any calculations to know the answer, but you can do calculations if it helps you.)

9. (4 pts) Let $B = \{(1, -1), (4, -5)\}$ be a basis for $\mathbb{R}^2$, and let $\vec{x} = (2, 1)$. Find the coordinate vector $[\vec{x}]_B$, the coordinate vector of $\vec{x}$ relative to B.

10. Define the linear transformation $T\colon P_2 \to \mathbb{R}^2$ by $T(\mathbf{p}) = (\mathbf{p}(0), \mathbf{p}(1))$, where $\mathbf{p}(a)$ is the number found by evaluating the polynomial $\mathbf{p}(x)$ at $x = a$.

   (a) (2 pts) What is $T(2 + 3x)$?

   (b) (4 pts) Is T a linear transformation? Explicitly check the conditions for a linear transformation.

   (c) (4 pts) Let $E = \{1, x, x^2\}$ be a basis for P_2. Find a matrix A representing T relative to the basis E (in other words, a matrix A so that if $\vec{v}$ is a coordinate vector for a polynomial $\mathbf{p}$, then $A\vec{v}$ is a coordinate vector for $T(\mathbf{p})$).

(d) (4 pts) Is T bijective? Completely justify your answer.

(e) Extra credit: (2 pts) Find the matrix $[T]_{B,E}$ for T relative to the basis $B = \{1,\, 2+3x,\, 1-x-x^2\}$ for P_2 and the standard basis for $\mathbb{R}^2$. In other words, find a matrix A so that if $\vec{v}$ is a coordinate vector for $\mathbf{p}$ relative to the basis B, then $A\vec{v}$ is $T(\mathbf{p})$.