Answer each question. Show all work.

1. Illustrate the following theorem by showing it works for matrices with at least 2 rows: If $A$ and $B$ are square matrices, then $\det(AB) = \det(BA)$.

2. Illustrate with matrices with at least 2 rows the theorem that a square matrix $A$ is noninvertible if and only if the matrix has a zero eigenvalue. Hint: you need to construct examples of two things—a noninvertible matrix with a zero eigenvalue, and an invertible matrix which has all nonzero eigenvalues. Show all work verifying each statement.
3. Let $V = \{(x, y) \text{ where } x, y \in \mathbb{R}\}$ be a set of “vectors.” Define vector addition by $(a, b) \oplus (c, d) = (a + c + 1, b + d + 2)$ and scalar multiplication by $k(a, b) = (ka + k - 1, kb + 2k - 2)$.

(a) Does this set have a “zero vector”? If so, what is it (show explicitly that your answer has the necessary property to be a zero vector)?

(b) Does every vector have an additive inverse? If so, what is the additive inverse of a vector $(a, b)$? If not, give an example of a vector that does not have an additive inverse.

(c) Does scalar multiplication distribute over vector addition: $k(\vec{u} \oplus \vec{v}) = k\vec{u} \oplus k\vec{v}$? If so, show it explicitly for a general scalar $k$ and general vectors $(a, b)$ and $(c, d)$. If not, give an explicit example where this property does not hold.
4. Let $V$ be the set of all polynomials of the form $ax^3 + bx$, where $a$ and $b$ are any real numbers, and assume we have normal polynomial addition and scalar multiplication. Is $V$ a vector space? If so, give a basis for $V$ (and explicitly show it is a basis for $V$ using the definition of “basis”) and give the dimension of $V$. If not, explain why $V$ is not a vector space.

5. Let $V$ be the set of all 2 by 2 matrices with determinant 1, and assume we have normal matrix addition and scalar multiplication. Is $V$ a vector space? If so, give a basis for $V$ (and explicitly show it is a basis for $V$ using the definition of “basis”) and give the dimension of $V$. If not, explain why $V$ is not a vector space.
6. Let $V$ be the vector space of 3 by 3 matrices that have the form

$$
\begin{bmatrix}
0 & a & b \\
-a & 0 & c \\
-b & -c & 0
\end{bmatrix},
$$

where $a$, $b$, and $c$ are real numbers.

(such matrices have the property that $A^T = -A$, and are called skew-symmetric matrices). Assume normal addition and scalar multiplication. Let

$$
A = \begin{bmatrix}
0 & 2 & 1 \\
-2 & 0 & 3 \\
-1 & -3 & 0
\end{bmatrix}, \quad B = \begin{bmatrix}
0 & -1 & 5 \\
1 & 0 & 2 \\
-5 & -2 & 0
\end{bmatrix}, \quad C = \begin{bmatrix}
0 & 5 & -2 \\
-5 & 0 & -1 \\
2 & 1 & 0
\end{bmatrix}, \quad D = \begin{bmatrix}
0 & 10 & -3 \\
-10 & 0 & -7 \\
3 & 7 & 0
\end{bmatrix}.
$$

Give a basis for $V$ from the matrices above, and write the coordinates of the remaining matrices relative to that basis. Justify your answer. Hint:

$$
\begin{bmatrix}
2 & -1 & 5 & 10 & 10 & 15 \\
1 & 5 & -2 & -17 & -3 & -5 \\
3 & 2 & -1 & 1 & -7 & -8
\end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix}
1 & 0 & 0 & 3 & -2 & -2 \\
0 & 1 & 0 & -4 & 1 & 1 \\
0 & 0 & 1 & 0 & 3 & 4
\end{bmatrix}
$$