Perfect Sequences of mth Roots of Unity

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Notation

Finite sequences of length n, $[a_0, a_1, \ldots, a_{n-1}]$ such that $a_j^m = 1$ for all j.

Particularly interested in $m \in \{2, 3, 4, 6\}$.

Autocorrelation

Cyclic Autocorrelation

$$\gamma_k := \sum_{j=0}^{n-1} \overline{a_j}\, a_{j+k}$$

(with $j + k$ taken $\mod n$.)

Acyclic Autocorrelation

$$c_k := \sum_{j=0}^{n-k-1} \overline{a_j}\, a_{j+k}$$

Necessary Conditions

- p^nth roots of unity cancel in size p cosets, so a perfect sequence must be of size kp for some $k \in \mathbb{N}$.

- $|a_0 + a_1 + \cdots + a_{n-1}|^2 = n$ means n factors as $A\overline{A}$ for some $A \in \mathbb{Z}[\omega]$.

Results from Turyn (1968)

A perfect sequence can be constructed

1. Of length m^2 using mth roots of unity.

   $$[0 \cdot 0, \ldots, 0 \cdot (m-1), 1 \cdot 0, \ldots, 1 \cdot (m-1), \ldots, (m-1) \cdot (m-1)]$$

2. Of length m using mth roots, if $m = p^r$, p an odd prime.

   $$[0^2, 1^2, \ldots, (m-1)^2]$$

3. If length n_1 and n_2 exist and are relatively prime, then length $n_1 \cdot n_2$ exists, using roots lcm(m_1, m_2). Constructed by pointwise dot product.

Our Results

A perfect sequence of length 2^{2k-1} using 2^kth roots of unity

Example: A sequence of length 8 using 16th roots of unity.

$$[0^2, 1^2, \ldots, 7^2] = [0, 1, 4, 9, 0, 9, 4, 1]$$

Therefore perfect sequences of all lengths exist.

This also gives us the obvious $[1, i]$ perfect sequence with quartic roots of unity.

Computational Results

Length	Root of Unity	Number
2	4	
3	3	6
4	2	4
5	5	20
6	12	12
7	7	42
8	4	32
9	3	54
10	≤ 20	
11	11	
12	6?	
13	13	
14	≤ 28	
15	≤ 15	
16	4	
17	17	
18	≤ 12	
19	19	
20	≤ 10	

- The algorithm found a perfect sequence of length 8 with quartic roots of unity. In general, is there a sequence of length p^3 using p^2 roots?

- For all examples of sequences of length n using mth roots of unity, we noticed

  $$\gcd(n, m) = \min(n, m).$$

  In the case m a prime, this is true. Is this always the case?

- Does there exist an example of a perfect sequence of length n using mth roots of unity where $n > m^2$?

Obtaining slides and program

Slides and documented C program available at `http://math.byu.edu/~grout/msri`.