The Minimum Rank Problem for Finite Fields

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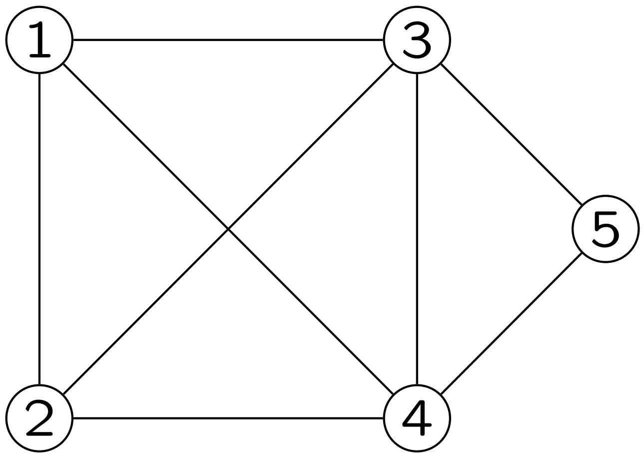
References

Barrett, van der Holst, Loewy, Graphs whose Minimal Rank is Two, Electronic Journal of Linear Algebra, volume 11 (2004), pp. 258–280

Barrett, van der Holst, Loewy, Graphs whose Minimal Rank is Two, Electronic Journal of Linear Algebra, volume 11 (2004), pp. 258–280

Barrett, Grout, March, The Minimal Rank Problem over a Finite Field, in preparation.

Example of $S(F,G)$



$$\Longleftrightarrow \begin{bmatrix} d_1 & * & * & * & 0 \\ * & d_2 & * & * & 0 \\ * & * & d_3 & * & * \\ * & * & * & d_4 & * \\ 0 & 0 & * & * & d_5 \end{bmatrix}$$

$$d_1, \ldots d_5 \in F.$$

Replace *s with any nonzero elements of F.

Example: Computing min rank in $\mathbb{R}, F_2, F_3$

$F = \mathbb{R}, F_3$:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 2 & 2 & 1 \\ 1 & 1 & 2 & 2 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

rank $A = 2$, so mr$(\mathbb{R}, G) = 2$ and mr$(F_3, G) = 2$.

But in F_2, $2 = 0$, so $A \notin S(F_2, G)$.

Example: Computing min rank in F_2

$F = F_2$:

Any $A \in S(F_2, G)$ has form $\begin{bmatrix} d_1 & 1 & 1 & 1 & 0 \\ 1 & d_2 & 1 & 1 & 0 \\ 1 & 1 & d_3 & 1 & 1 \\ 1 & 1 & 1 & d_4 & 1 \\ 0 & 0 & 1 & 1 & d_5 \end{bmatrix}$.

$A[145|235] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & d_5 \end{bmatrix}$ has determinant 1 so rank $A \geq 3$.

Therefore $\text{mr}(F_2, G) \geq 3$.

Idea of Algorithm

To find the graphs characterizing $\{G \mid \mathrm{mr}(F,G) \leq k\}$:

1. Construct all matrices A of rank $\leq k$ over F. Use the fact
$$A = U^t B U \iff \mathrm{rank}(A) \leq k.$$
(B is $k \times k$, rank k; U is $k \times n$.)

2. Return non-isomorphic graphs corresponding to matrices.

Problem: Too many matrices.

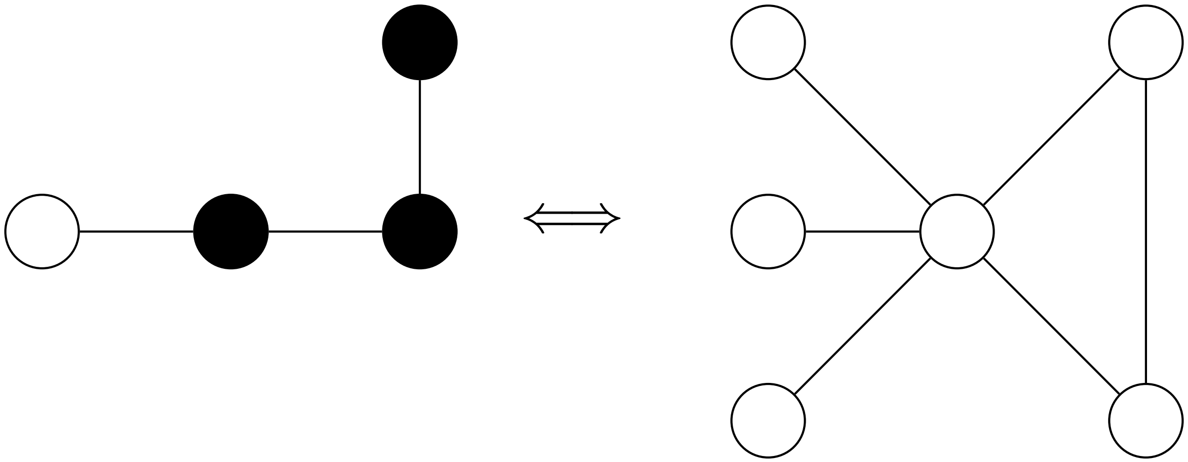
Solution: We only need the zero-nonzero patterns for the matrices. Be smarter by understanding $A = U^t B U$ better.

$$A = U^t B U$$

Feature/operation on U	**Effect on graph corresponding to A**
Column in U	vertex in graph
Column in U isotropic wrt B	zero entry for the vertex on diagonal
Column in U not isotropic wrt B	nonzero entry for the vertex on diagonal
Two columns orthogonal wrt B	no edge between corresponding vertices (zero matrix entry)
Two columns not orthogonal wrt B	edge between corresponding vertices (nonzero matrix entry)

$$A = U^t B U$$

Feature/operation on U	**Effect on graph corresponding to A**
Duplicate columns (isotropic)	independent set, vertices have same neighbors
Duplicate columns (non-isotropic)	clique, vertices have same neighbors
Columns multiples of each other	corresponding vertices have same neighbors (remember, only the zero-nonzero pattern is needed, and there are no zero divisors in F)
Interchanging two columns	relabel vertices



- black vertex $\iff$ clique

- white vertex $\iff$ independent set

- edge $\iff$ all possible edges

- cliques or independent sets can be empty

Algorithm

To find the graphs characterizing $\{G \mid \text{mr}(F, G) \leq k\}$:

1. Find a maximal set of k-dimension vectors in F such that no vector is a multiple of any other. These are columns in U.

2. Construct all *interesting* matrices A of rank $\leq k$ over F. Use the fact

   $$A = U^t B U \iff \text{rank}(A) \leq k.$$

   (B is $k \times k$, rank k; U is $k \times n$.)

3. Return non-isomorphic *marked* graphs corresponding to matrices.

Characterizing marked graphs for F_3

Rank	Vertices	Edges
2	5	5
2	5	4
3	14	54
4	41	525
4	41	528
5	122	4 860
6	365	44 100
6	365	44 109
7	1094	398 034

Finding forbidden subgraphs

Let S be the set of marked subgraphs of our characterizing graphs.

For each (normal) graph G

1. Construct the set T of possible marked graphs for G (can do this in exponential time).

2. If $S \cap T \neq \emptyset$, then G is a substitution graph of the characterizing graphs.

3. If $S \cap T = \emptyset$, then G is forbidden.

Open Questions

- Given a finite field F and positive integer k, what is a good upper bound for the number of vertices in minimal forbidden subgraphs?

- Is the bound 8 for $F = F_2$ and $k = 3$?

- Let G be any graph and let F be a finite field, $\operatorname{char} F \neq 2$. Is $\operatorname{mr}(\mathbb{R}, G) \leq \operatorname{mr}(F, G)$? (true if $\operatorname{mr}(\mathbb{R}, G) \leq 3$).