USING VARIANTS OF ZERO FORCING TO BOUND THE INERTIA SET OF A GRAPH[*]

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Abstract. Zero forcing is a combinatorial game played on a graph with a goal of changing the color of every vertex at minimal cost. This leads to a parameter known as the zero forcing number that can be used to give an upper bound for the maximum nullity of a matrix associated with the graph. We introduce a variation on the zero forcing game that can be used to give an upper bound for the maximum nullity of such a matrix when it is constrained to have exactly q negative eigenvalues.

This constrains the possible inertias that a matrix associated with a graph can achieve and gives a method to construct lower bounds on the inertia set of a graph (which is the set of all possible pairs (p, q) where p is the number of positive eigenvalues and q is the number of negative eigenvalues).

Key words. Zero forcing, Inertia sets; Isotropic subspaces.

AMS subject classifications. 05C50, 15A03.

1. Introduction.

An important property of a Hermitian matrix is its (partial) inertia (p, q), where p is the number of positive eigenvalues and q is the number of negative eigenvalues. We explore how the positions of zero/nonzero entries in a Hermitian matrix affect the inertia of the matrix. Given a simple graph G (i.e., undirected, no loops, no multiedges), we can associate with G a collection of Hermitian matrices $\mathcal{S}(G)$ such that for each matrix, an off-diagonal entry is non-zero if and only if the entry corresponds to an edge of G (there are no restrictions on the diagonal entries). The graph G represents the pattern of zero/nonzero entries in the matrices in $\mathcal{S}(G)$. The *inertia set* $\mathcal{I}(G)$ of G is the set of all inertias of matrices in $\mathcal{S}(G)$.

Clearly, $p + q \leq n$ for an $n \times n$ matrix. If $(p, q) \in \mathcal{I}(G)$, then $(p', q') \in \mathcal{I}(G)$ where $p' \geq p$, $q' \geq q$, and $p' + q' \leq n$ [4]. Thus the really interesting question is: what are the possible minimal inertias? Since the nullity of an n by n Hermitian matrix with inertia (p, q) is $n - p - q$, the maximum nullity of all matrices in $\mathcal{S}(G)$, denoted $M(G)$, gives the constraint that $p + q \geq n - M(G)$. However, calculating

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the maximum nullity $M(G)$ (or the minimum rank parameter $\mathrm{mr}(G) = n - M(G)$, equivalently) is difficult in general.

An upper bound for the maximum nullity of all graphs in $\mathcal{S}(G)$ is the zero forcing number of the graph, denoted $Z(G)$ (see [1, 2, 7]), which gives us the much easier to compute constraint $p + q \geq n - Z(G)$. The zero forcing number can be computed in a combinatorial coloring game of the graph. In this paper, we generalize the game that computes $Z(G)$ to a new game that computes $Z_q(G)$, an upper bound for the nullity of matrices in $\mathcal{S}(G)$ that have q negative eigenvalues. We will see that for some graphs G, $Z_q(G) < Z(G)$ for some q, which improves our understanding of the inertia set of G.

In Section 2 we will give a short review of both zero forcing and a variant known as positive semidefinite zero forcing, which we will generalize in Section 3 with our new zero forcing parameter $Z_q(G)$. We then will introduce $\widehat{Z}_q(G)$ in Section 4 which will give a further improvement for determining the inertia set of a graph. In Section 5 we will show how to use these new parameters to gain information about $\mathcal{I}(G)$. Finally, in Section 7 we give an algorithm for computing $Z_q(G)$.

We will use the following notation throughout this paper: for a subset of vertices $W \subseteq V$, let $G[W]$ be the induced subgraph of G on the vertices W.

2. Review of zero forcing and semidefinite zero forcing.

The zero forcing number of a graph is computed using the combinatorial coloring *zero forcing game*, which is played on a graph by a single player called Black. The game starts with all vertices colored white, and the goal is to color all vertices black while spending as few tokens as possible. There are two operations Black can perform: spending a token to color a vertex black, and applying a *color change rule*. The color change rule, given a graph with black and white vertices, is to select a white vertex v that is the unique white neighbor of a black vertex u, and change v from white to black. The vertex v is said to be *forced* by u.

> **Zero Forcing Game** – All the vertices of the graph G are initially colored white and there is one player, known as Black, who has tokens. Black will repeatedly apply one of the following two operations until all vertices are colored black:
>
> 1. For one token, Black can change any vertex from white to black.
> 2. At no cost, Black can apply the color change rule on the entire graph.

The minimum number of tokens that Black must use in a given strategy to change all of the vertices from white to black is the zero forcing number $Z(G)$. Given any sequence of operations in this game, Black can always reorder the moves so that all of the token-spending happens before any color change rules are applied. In this setting the set of vertices that Black initially spends his tokens on are known as a *zero forcing set* and $Z(G)$ is then the minimal size of a zero forcing set. This is the way that zero forcing is usually defined and introduced (i.e., as a set rather than a strategy in a game).

The motivation behind the definition of the color change rule comes from looking at vectors in the null space of a matrix. Suppose $\mathbf{x}$ is in the null space for a matrix $A \in \mathcal{S}(G)$, i.e., we have $A\mathbf{x} = \mathbf{0}$, and that $\mathbf{x}$ is 0 for each vertex currently colored black. The color change rule is the observation that some additional entries of $\mathbf{x}$ might also be forced to be 0. Suppose vertex i is black and its only white neighbor is vertex j; then $(A\mathbf{x})_i = a_{ij}x_j = 0$. But since $a_{ij} \neq 0$ (i.e., the edge ij exists in G), then we must have $x_j = 0$. We mark this additional piece of information by coloring vertex j black.

By the color change rule it follows that any vector in the null space of $A \in \mathcal{S}(G)$ which is 0 on a zero forcing set of G is the $\mathbf{0}$ vector. On the other hand, by considering the dimensions of various subspaces we have the following.

Observation 2.1. *If the nullity of a matrix is more than k, then for any k specified entries, there is a nonzero vector $\mathbf{x}$ in the null space which will vanish at those specified entries.*

Proposition 2.2 (AIM [1]). *For any $A \in \mathcal{S}(G)$ the nullity is bounded above by the size of any zero forcing set, in particular by $Z(G)$.*

Proof. Suppose not, then by the above observation there would exist a nonzero vector in the null space which is 0 on all the vertices corresponding to a minimally sized zero forcing set. But this is impossible since the only vector in the null space which is 0 on a zero forcing set is $\mathbf{0}$. □

Another observation we will use later is that if there exists a null vector $\mathbf{x}$ with support on the white vertices, then the color change rule will not change the color of any vertex in the support of $\mathbf{x}$.

A modification of zero forcing was considered when the matrices in $\mathcal{S}(G)$ were further restricted to require that the matrices be positive semidefinite. Since this imposes additional relationships on entries in the matrix it is possible to modify the game to give Black more options to force vertices to be black.

> **Semidefinite Zero Forcing Game** – All the vertices of the graph G are initially colored white and there is one player, known as Black, who has tokens. Black will repeatedly apply one of the following three options until all vertices are colored black:

1. For one token, Black can change any vertex from white to black.
2. At no cost, Black can apply the color change rule on the entire graph.
3. Let the vertices currently colored black be denoted by B, and $W_1, W_2, \ldots, W_k$ be the vertex sets of the connected components of $G[V \setminus B]$. At no cost, Black can apply the color change rule on $G[B \cup W_i]$ for some $1 \leq i \leq k$.

The new option given by this game is the opportunity to apply the color change rule on a *smaller* part of the graph, i.e., we may be allowed to ignore some of the white neighbors of a black vertex, so that the "unique white neighbor" of a black vertex in the subgraph can be forced to black. The minimal number of tokens that Black must use to change all of the vertices from white to black in this game is denoted $Z_+(G)$. As before, Black can elect to initially only spend tokens and then apply either of the forcing options using the color change rule for the remainder of the game, and so the literature discusses positive semidefinite forcing sets and not positive semidefinite forcing strategies. In this setting $Z_+(G)$ is the size of the smallest possible such set.

Theorem 2.3 (Barioli et al. [2]). *The nullity of $A \in \mathcal{S}(G)$ when A is positive semidefinite is at most $Z_+(G)$.*

3. Zero forcing with q negative eigenvalues.

The semidefinite zero forcing number gives an indication of how to generalize zero forcing—we give Black the possibility of working with a smaller graph. This leads us to the general Z_q-forcing game which introduces a second player called White. The role of White is to limit the choice of smaller graphs on which Black is allowed to play. In particular, the goal of White is to maximize the number of tokens Black must spend to color the graph.

The minimal number of tokens that an optimal strategy for Black uses to change all of the vertices from white to black, regardless of the play of White, is the zero forcing number for matrices with q negative eigenvalues, denoted $Z_q(G)$.

Z_q-Forcing Game – All the vertices of the graph G are initially colored white and there are two players, known as Black (who has tokens) and White. Black will repeatedly apply one of the following three options until all vertices are colored black.

1. For one token, Black can change any vertex from white to black.
2. At no cost, Black can apply the color change rule on the entire graph G.
3. Let the vertices currently colored black be denoted by B, and $W_1, \ldots, W_k$ be the vertex sets of the connected components of $G[V \setminus B]$. Black selects *at least* $q + 1$ of the W_i and announces the selection to White. White then will select a nonempty sub-

set of these components, say $\{W_{i_1}, \ldots, W_{i_\ell}\}$ (with $\ell \geq 1$) and announces it back to Black. At no cost, Black can apply the color change rule on $G[B \cup W_{i_1} \cup \cdots \cup W_{i_\ell}]$.

Unlike zero forcing and positive semidefinite zero forcing, it may now be the case that no optimal strategy for Black allows spending all the tokens up front. In other words there are graphs where Black will vary the choice of where to spend tokens depending on the response of White. Instead of Z_q-forcing sets, there are Z_q-forcing strategies.

Example 3.1. *Consider the graph shown in Figure 3.1a on* 9 *vertices. We show a Z_1-forcing strategy for Black that uses* 4 *tokens to make all the vertices of the graph black. First, Black spends two tokens to color the vertices* 5 *and* 8 *black which in the next round will force* 1 *and* 9 *at no cost. Black now picks a vertex from* $\{2,3,4\}$ *and a vertex from* $\{6,7\}$ *and declares the two vertices to White. Whatever is returned is then forced and this continues until one of the two sets has been completely colored black. In the worst case scenario for Black, White will have only returned the vertices from* $\{6,7\}$ *in which case the graph is as shown in Figure 3.1c. At this point Black can spend at most two tokens to get all but one of the remaining vertices to become black and the last vertex will then be switched to black at no cost by the color change rule.*

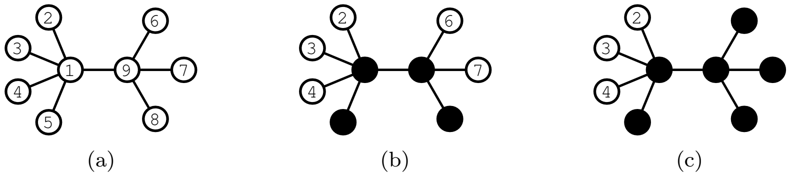

Fig. 3.1: An example of a graph with $Z_1(G) = 4$.

On the other hand, if Black had chosen to initially spend four tokens before using any free forcing, then at least two of $\{2,3,4,5\}$ *or two of* $\{6,7,8\}$ *would not be black. White could now protect a pair of pendant vertices with a common neighbor and Black would not be able to color the entire graph.*

This example also shows that White should not always return all the subsets Black has declared, even though intuitively it would seem that the more white vertices there are, the more difficult it should be for Black to apply the color change rule. (The important point is that the intuition holds only when the white components are "close" in the graph.)

Proposition 3.2. *For any graph G on n vertices, we have*

$$Z_+(G) = Z_0(G) \leq Z_1(G) \leq Z_2(G) \leq \cdots \leq Z_n(G) = Z(G).$$

Proof. For Z_0, the third operation reduces to the semidefinite zero forcing operation because once Black has declared one component, White has no choice but to return it.

Suppose that t and s are integers such that $t \leq s$. Black can now use the strategy for the Z_s-forcing game that uses at most $Z_s(G)$ tokens in the Z_t-forcing game and force all the vertices to be black. It follows that $Z_t(G) \leq Z_s(G)$.

Finally, Black cannot invoke the third operation of the Z_q game when there are fewer than $q+1$ connected components in $G[V \setminus B]$, which is always the case when $q = n$ (and in most graphs is always the case for several smaller values of q, as well). Since the sequence $Z_q(G)$ converges to $Z(G)$ for all graphs, we may think of the zero forcing number of a graph as $Z_\infty(G)$. □

We are now ready to state our main result.

Theorem 3.3. *The nullity of $A \in \mathcal{S}(G)$ when A has exactly q negative eigenvalues is at most $Z_q(G)$.*

The proof of Theorem 3.3 can be easily modified to establish the stronger result for matrices with at most q negative eigenvalues. The proof will make use of *isotropic subspaces*. An *isotropic subspace* of a square matrix is a vector subspace where $\mathbf{x}^*A\mathbf{x} = 0$ for all $\mathbf{x}$ in the subspace.

Theorem 3.4 (Gohberg et al. [5, Theorem 1.5]). *The maximum possible dimension of an isotropic subspace for an $n \times n$ Hermitian matrix A is $n - p - q + \min\{p, q\}$ where p and q are the number (counting multiplicity) of positive and negative eigenvalues respectively.*

Corollary 3.5. *For a Hermitian matrix A, let R be an isotropic subspace of dimension more than $\min\{p, q\}$ where p and q are the number (counting multiplicity) of positive and negative eigenvalues respectively. Then R contains a nonzero vector in the null space.*

Proof. Assume there is no nonzero null vector in R. A basis for the nullspace of A has $n - p - q$ vectors, and $\mathbf{x}^*A\mathbf{x} = 0$ for each of these vectors. These vectors together with a basis for R form a basis for an isotropic subspace that has more than $n - p - q + \min\{p, q\}$ vectors, in contradiction to Theorem 3.4. □

Proof. [Proof of Theorem 3.3] Let A be a matrix with q negative eigenvalues and nullity m. We will use the matrix to produce a strategy for White in the Z_q-forcing game that will force black to spend at least m tokens. This will establish $m \leq Z_q(G)$.

The game proceeds as follows: Every time Black spends a token to color a vertex black, White considers the subspace of the null space with support contained in the white vertices and constructs a null vector $\mathbf{x}$ that is nonzero with the same support (such a vector exists by general position). We now show that $\mathbf{x}$ can be used to protect the support of the null space.

Recall from the discussion following the color change rule in Section 2 that the only time a vertex v changes from white to black is when, for any null vector $\mathbf{x}$ whose support is a subset of the white vertices, the support of $\mathbf{x}$ is in fact a strict subset because the entry corresponding to v has to be zero. When applying the color change rule to the entire graph, Black cannot change the color for anything in the support of $\mathbf{x}$.

Now consider the option where White is handed several components. Suppose that B are the vertices colored black and $W_1, W_2, \ldots, W_k$ are the vertices of the maximally connected components of $G[V \setminus B]$. Then by appropriate relabeling we can assume that

$$A = \left[\begin{array}{c|c|c|c|c|c} A_1 & O & O & \cdots & O & B_1^* \\ \hline O & A_2 & O & \cdots & O & B_2^* \\ \hline O & O & A_3 & \cdots & O & B_3^* \\ \hline \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline O & O & O & \cdots & A_k & B_k^* \\ \hline B_1 & B_2 & B_3 & \cdots & B_k & C \end{array}\right] \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \\ \vdots \\ \mathbf{x}_k \\ \mathbf{0} \end{bmatrix},$$

where A_i is the submatrix on $G[W_i]$ and C is the submatrix on $G[B]$. For any index i, since $A\mathbf{x} = \mathbf{0}$, then $A_i\mathbf{x}_i = \mathbf{0}$. Let $\widehat{\mathbf{x}}_i$ be equal to $\mathbf{x}$ on the ith component and zero elsewhere.

If $\mathbf{y} = \sum a_i \widehat{\mathbf{x}}_i$ is some linear combination of $\widehat{\mathbf{x}}$, we have

$$\mathbf{y}^* A \mathbf{y} = \mathbf{y}^* \begin{bmatrix} \vdots \\ a_i A_i \mathbf{x}_i \\ \vdots \\ \sum a_i B_i \mathbf{x}_i \end{bmatrix} = \begin{bmatrix} \cdots & a_i \mathbf{x}_i & \cdots & \mathbf{0} \end{bmatrix} \begin{bmatrix} \vdots \\ \mathbf{0} \\ \vdots \\ \sum a_i B_i \mathbf{x}_i \end{bmatrix} = 0.$$

This shows that the vectors $\widehat{\mathbf{x}}_1, \widehat{\mathbf{x}}_2, \ldots, \widehat{\mathbf{x}}_k$ span an isotropic subspace of A. Black has selected at least $q + 1 > \min\{p, q\}$ of the W_i, and each W_i is associated with $\widehat{\mathbf{x}}_i$. Let W be the set of vectors $\widehat{\mathbf{x}_\mathbf{i}}$ corresponding to the components Black has selected.

If any $\widehat{\mathbf{x}}_i \in W$ is zero, then White will return the corresponding single component to Black since any forcing on that component will not change a vertex in the support of $\mathbf{x}$.

On the other hand, if none of the $\widehat{\mathbf{x}}_i \in W$ are zero, then they are independent and by Corollary 3.5 there is a nontrivial null vector $\mathbf{z}$ in the subspace spanned by W, $\mathbf{z} = \sum b_i \widehat{\mathbf{x}}_i$. White returns all of the W_i components for which $b_i \neq 0$. Since $\mathbf{z}$ is a null vector with support contained in the white vertices handed back to Black, Black cannot force a vertex in the support of $\mathbf{z}$ to black. On the white vertices handed back, the support of $\mathbf{z}$ is the same as the support of $\mathbf{x}$, so Black cannot force any vertex in the support of $\mathbf{x}$ to black.

In both operations of applying the color change rule we see that Black cannot change the color of a vertex in the support of $\mathbf{x}$. The only way Black can change the color of any vertex in the support of $\mathbf{x}$ is to spend a token. Each time a token is spent this adds one new constraint to the set of null vectors that still have support on the white vertices, i.e., it reduces the dimension of the restricted null space by at most one. We started with a dimension of m and to be able to have successfully colored we must get to a dimension of 0 which means that Black has to spend at least m tokens. □

The proof shows that any matrix of nullity m gives White a strategy that forces Black to spend at least m tokens, but not every strategy comes from a matrix. If there is a strategy for White that outperforms every matrix-based strategy, the bound will not be tight.

4. The parameter $\widehat{Z}_q(G)$.

On all but one tree with 10 or fewer vertices, $Z_q(G)$ is a tight bound for the maximum nullity of a matrix associated with the tree that has q negative eigenvalues. The one exception is shown in Figure 4.1, also known as the Barioli-Fallat Tree.

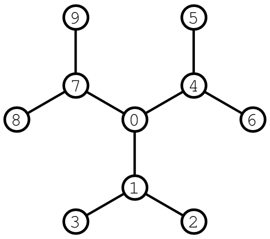

Fig. 4.1: The Barioli-Fallat Tree T.

For this tree, the maximum nullity for a matrix with 1 negative eigenvalue is 2 (a method to compute the inertia set of any tree is given in [4]). However, $Z_1(T) = 3$. This can be seen by noting that regardless of where Black spends the first two tokens there will be a pair of pendant white vertices which share a common neighbor left in the graph. White can simply protect that pair in its responses, forcing Black to spend a third time; conversely if Black spends on $\{3, 6, 9\}$, then the rest of the graph is easily forced.

Recently an improvement for zero forcing, denoted $\widehat{Z}(G)$, was introduced by Barioli et al. [3] which uses information about the diagonal. We can use the same approach to improve $Z_q(G)$ to $\widehat{Z}_q(G)$, which gives the correct bounds for the Barioli-Fallat tree and improved bounds for many other graphs.

The idea is to introduce auxiliary graphs $\widehat{G}$ which are the same as G except each vertex is either looped or unlooped. The color change rule is extended to a *looped color change rule*: any vertex with exactly one white neighbor can force the neighbor to change to black. A looped vertex is a neighbor to itself, so it can force itself from white to black provided all other neighbors are black. An unlooped (black or white) vertex is not a neighbor to itself, so if it has exactly one neighbor which is white then it will force that neighbor to be black regardless of its own color.

With this new convention for applying the color change rule, we define

$$\widehat{Z}_q(G) = \max_{\widehat{G} \in \mathcal{G}} Z_q(\widehat{G})$$

where the maximum runs over the set of all possible auxiliary graphs $\mathcal{G}$.

By adding loops and unloops, we are effectively specifying the zero/nonzero pattern on the diagonal of associated matrices, i.e., by putting loops at nonzero entries on the diagonal and unloops at zero entries on the diagonal. Since we are taking the maximum over all possible diagonal zero/nonzero patterns, this still is an upper bound for the nullity of matrices in $\mathcal{S}(G)$. In practice, we can use the symmetry of the graph (i.e., the automorphism group) and careful analysis of when the looped color change rule is applied to avoid having to consider all possible diagonal patterns.

We now show that $\widehat{Z}_1(T)$ is a tight bound for the Barioli-Fallat tree T. Suppose that there was a loop at 2. Now consider the following strategy for Black. Black spends at 6 and 8; forcing then changes 4 and 7 to black; Black now hands White $\{5\}$ and $\{9\}$; whatever is returned is forced and forcing on the entire graph gets us to only the vertices 2 and 3 as white; 2 is looped so it now forces itself to black; forcing then gives 3. In this case Black is able to color all the vertices black using only two tokens. By symmetry the same strategy works if there is any loop at a leaf.

Now consider a strategy for Black when all of the leaves are unlooped. Without spending, Black has that $1, 4, 7$ are black by the unlooped leaves; Black hands White

$\{2\}$ and $\{6\}$ and whatever is returned is forced, without loss of generality let us suppose that only $\{2\}$ is returned; Black now hands White $\{6\}$ and $\{8\}$ and whatever is returned is forced, without loss of generality let us suppose that only $\{6\}$ is returned; Black spends on 0 and 8; the remaining vertices are then forced.

All possibilities of being looped and unlooped fall into one of these two cases, and in each case, Black only needed to spend at most 2, showing that $\widehat{Z}_1(T) = 2$.

Since Black can ignore the information about the looped and unlooped vertices, we have $\widehat{Z}_q(G) \leq Z_q(G)$, so $\widehat{Z}_q(G)$ is a better bound. For trees, the parameter $\widehat{Z}_q(G)$ correctly gives the right maximum nullity for all but one tree with 16 or fewer vertices. The one exception is a graph on 16 vertices shown in Figure 4.2, which is a Barioli-Fallat tree with a pendant vertex added to each original pendant vertex.

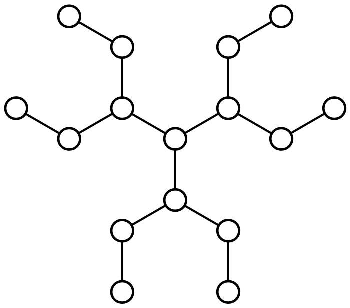

Fig. 4.2: The extended Barioli-Fallat Tree

5. Finding lower bounds for inertia sets. The motivation for $Z_q(G)$ was to provide lower bounds for inertia sets. Recall that the inertia set of a graph, $\mathcal{I}(G)$, is the set of all possible inertias (p, q) for matrices in $\mathcal{S}(G)$, where p is the number of positive eigenvalues and q is the number of negative eigenvalues.

Observation 5.1. *For a graph G on n vertices, $(n - q - Z_q(G) - 1, q) \notin \mathcal{I}(G)$.*

This follows since $p = n - q - m$ where m is the nullity of the matrix, since $Z_q(G)$ is an upper bound on the nullity we can conclude that $p \geq n - q - Z_q(G)$. This leads to the conclusion that if we have q negative eigenvalues then we cannot have $n - q - Z_q(G) - 1$ or fewer positive eigenvalues, i.e., these points are not in the inertia set of the graph.

Since (p, q) is in the inertia set if and only if (q, p) is in the inertia set (i.e., $A \in \mathcal{S}(G)$ if and only if $-A \in \mathcal{S}(G)$) then the above argument also shows that if we have p positive eigenvalues then we cannot have $n - p - Z_p(G) - 1$ or fewer negative eigenvalues, i.e., these points are also not in the inertia set of the graph.

To eliminate points in the inertia set, we simply compute $Z_q(G)$ over a range of values. As an example let us consider the Desargues graph on 20 vertices, shown (with various sets marked) in Figure 5.1. We will consider variations of the games discussed and what information we can gain about the inertia set. Each variation will eliminate points from our possible inertia set. These improvements are summarized in Figure 5.2, where each invariant is used to label the points that it progressively eliminates. The filled circles are the inertia set of the graph.

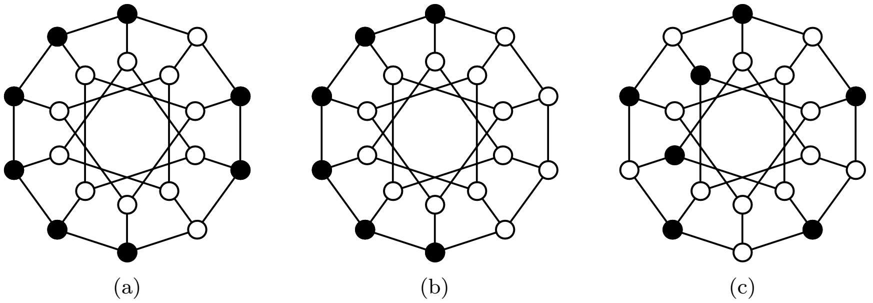

Fig. 5.1: Several forcing sets for various games on the Desargues graph.

(a) By Proposition 2.2 we have that the maximum nullity of a matrix associated with the graph is bounded by $Z(G)$. For the Desargues graph, $Z(G) = 8$ (a forcing set is shown in Figure 5.1a). This shows that we must have $p + q \geq 12$.

(b) In addition to the above work, we can use the Z_+ game. We have $Z_+(G) = 6$ (a forcing set is shown in Figure 5.1b), so by Theorem 2.3 we conclude $(12, 0)$, $(0, 12)$, $(13, 0)$ and $(0, 13)$ cannot be in the inertia set.

(c) We can work with Z_q instead, but insist that Black must always spend tokens up front before forcing. This restriction does not give Z_q in general since the order of moves matters in the Z_q game. In the case of the Desargues graph, Black can spend only 7 tokens to color the graph in the Z_1 game (see Figure 5.1c), and this is the best Black can do if spending moves are only allowed at the start of the game. This shows that $(11, 1)$ and $(1, 11)$ are also not points in the inertia set.

(d) We can also play the (unrestricted) Z_q game for all q to get

$$Z_0(G) = Z_1(G) = 6 < Z_2(G) = 7 < Z_3(G) = \ldots = 8.$$

This shows that $(12, 1)$, $(1, 12)$, $(10, 2)$ and $(2, 10)$ are also not in the inertia set.

(e) Applying the $\widehat{Z}_q$ game, we have

$$\widehat{Z}_0(G) = \cdots = \widehat{Z}_5(G) = 6 < \widehat{Z}_6(G) = \cdots = 8.$$

This shows that $(6, 6)$, $(6, 7)$ and $(7, 6)$ are the only possible points in the inertia with $p + q \leq 13$.

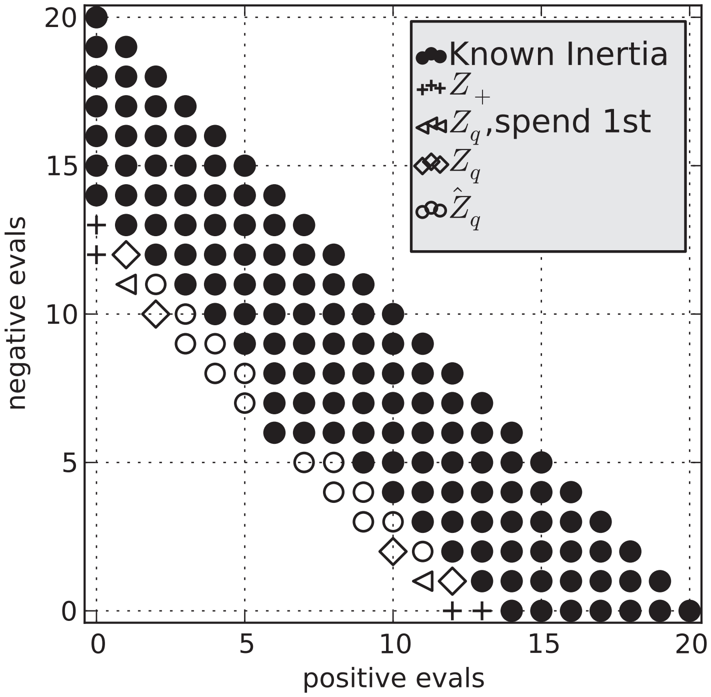

Fig. 5.2: Inertia refinements for various games on the Desargues graph.

We have now computed lower bounds for the inertia set, the best coming from $\widehat{Z}_q$. These bounds are tight. First, we can produce some specific matrices showing that $(6,6)$ and $(14,0)$ are in the inertia set. For the inertia $(6,6)$, we construct a $0,1,-1$ matrix in $\mathcal{S}(G)$ by puttings 0s on the diagonal and 1s for the edges except for five edges which receive -1: these five edges are every other spoke between the inner and outer cycles. The eigenvalues for the resulting matrix are $[-\sqrt{5}]^6, [0]^8, [\sqrt{5}]^6$ (where exponents denote multiplicity). Further if we add $\sqrt{5}I$, then the resulting matrix has eigenvalues $[0]^6, [\sqrt{5}]^8, [2\sqrt{5}]^6$, showing that $(14,0)$ is in the inertia set. A longer calculation considering the Desargues graph as the bipartite double cover of the Petersen graph can be used to produce matrices of all the remaining inertias (p,q) with $p+q=14$.

6. Other uses of Z_q and $\widehat{Z}_q$. We note two ways to use Z_q and $\widehat{Z}_q$ to obtain more information about a graph G.

Using $\widehat{Z}_q$ to determine zero/nonzero patterns. The diagonal of the matrix that we constructed to show that $(6, 6)$ was in the inertia set for the Desargues graph was all 0s. Our computations on $\widehat{Z}_6(G)$ indicate that for this graph, *every* matrix with inertia $(6, 6)$ must have 0s on the diagonal. In other words, in the computation of $\widehat{Z}_6(G)$, we actually compute a bound for each zero/non-zero pattern on the diagonal. For this case, we saw that any matrix with a nonzero on the diagonal (i.e., at least one vertex had a loop) had $\widehat{Z}_6(G) \leq 7$, and so could not have inertia $(6, 6)$. This same approach can be used with other graphs to give restrictions for zero/non-zero patterns of the diagonal for some points in the inertia set.

Disjoint union of graphs. If G is the disjoint union of G_1 and G_2, then the inertia set for G is exactly the Minkowski sum of the inertia sets of G_1 and G_2, i.e., $\{(x_1 + x_2, y_1 + y_2) \mid (x_1, y_1) \in \mathcal{I}(G_1), (x_2, y_2) \in \mathcal{I}(G_2)\}$, since a matrix for G is just the block diagonal sum of a matrix for G_1 and a matrix for G_2.

Since Z_q helps us exclude points from the inertia set, this suggests that we should use Z_q to compute sets of points that are not in the inertia set for G_1 and G_2 respectively, knowing that the Minkowski sum will give us points not in the inertia set of G. There is a computational advantage in computing Z_q on the smaller graphs G_1 and G_2 rather than directly computing $Z_q(G)$.

In some cases, we will also get better information by using $Z_q(G_1)$ and $Z_q(G_2)$ in this manner. For example consider the graph T shown in Figure 6.1.

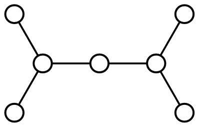

Fig. 6.1: A tree with $Z_0(T) = 1$, $Z_1(T) = 2$, and $Z_2(T) = 3$.

By taking the Minkowski sum of points excluded from the inertia by $Z_q(T)$, we conclude that matrices for $T \sqcup T$ with 2 negative eigenvalues have maximum nullity most 4. However $Z_2(T \sqcup T) = 5$, which is a worse bound.

In general, by similar reasoning, we have

$$\max_{s+t=q} \big(Z_s(G) + Z_t(H)\big) \leq Z_q(G \sqcup H),$$

and the example above shows that we can have a strict inequality.

7. Algorithmic implementation for computing Z_q. We finally turn to the issue of efficiently computing $Z_q(G)$. The goal is to find the minimal cost (i.e., minimal times option (1) is applied) necessary to force the entire graph to be colored black. Given a subset of vertices $U \subseteq V$ which we assume is already colored black, we determine the minimal cost, denoted $\text{cost}(U)$, necessary to force the remaining vertices black using the rules of the Z_q game. The trivial cases are $\text{cost}(V) = 0$ and $\text{cost}(\emptyset) = Z_q(G)$.

We work backwards from large U to small U. At each stage we consider each of the three options available to Black and choose the option which minimizes cost. Since forcing is always free, we continually carry out that option (this reduces the amount of computation and storage). The procedure is shown in Algorithm 1, a variation of which has been implemented in Sage [8], and is publicly available online [6]. In the algorithm, we let $F(G, B)$ be the set of vertices in G that can be colored black by repeatedly applying the color change rule with the vertices in B initially colored black.

```
input  : A graph $G = (V, E)$ and parameter $q$
output: The value $Z_q(G)$
cost$(V) \leftarrow 0$;
for $i \leftarrow |V| - 1$ to $0$ do
    foreach $U \subseteq V$ with $|U| = i$ and $F(G, U) = U$ do
        $b, c,$ cost$(U) \leftarrow \infty$;
        let $K$ be the sets of vertices of the connected components of $G \setminus U$;
        foreach $J \subseteq K$ with $|J| = q + 1$ do
            $b' \leftarrow -\infty$;
            foreach $I \subseteq J$ with $I \neq \emptyset$ do
                $b' \leftarrow \max\big(b', \texttt{cost}(F(G, F(G[U \cup I], U)))\big)$;
            end
            $b \leftarrow \min(b, b')$;
        end
        foreach $v \in V \setminus U$ do
            $c \leftarrow \min(c, \texttt{cost}\big(F(G, U \cup \{v\})\big) + 1)$;
        end
        cost$(U) \leftarrow \min(b, c)$;
    end
end
return cost$(\emptyset)$;
```

Algorithm 1: The procedure $Z(G, q)$ to calculate the zero forcing numbers

The cost function generated by this algorithm can be used by Black to determine a strategy for playing that uses at most $Z_q(G)$ tokens. At each stage, Black chooses an option that will allow a win with the number of tokens available. Also, Black will be able to win by spending all the tokens up front if and only if there is some $U \subseteq V$ with $|U| = Z_q(G)$ and $\operatorname{cost}(U) = 0$.

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