The Minimum Rank Problem for Finite Fields

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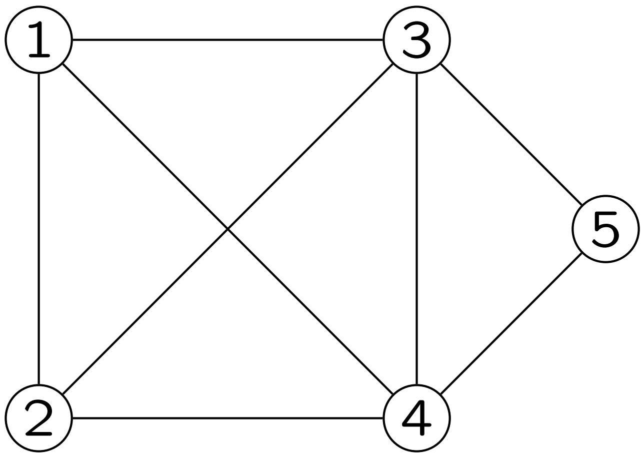
Correspondence of G and matrices



$$\Longleftrightarrow \begin{bmatrix} d_1 & * & * & * & 0 \\ * & d_2 & * & * & 0 \\ * & * & d_3 & * & * \\ * & * & * & d_4 & * \\ 0 & 0 & * & * & d_5 \end{bmatrix}$$

$d_1, \ldots d_5 \in F$.

Replace the *s with any nonzero elements of F.

$\text{mr}(F, G) =$ minimum rank of corresponding matrices.

Example: Computing min rank in $\mathbb{R}, F_2, F_3$

$F = \mathbb{R}, F_3$:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 2 & 2 & 1 \\ 1 & 1 & 2 & 2 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

rank $A = 2$, so mr$(\mathbb{R}, G) = 2$ and mr$(F_3, G) = 2$.

But in F_2, $2 = 0$, so A doesn't correspond to G.

Example: Computing min rank in F_2

$F = F_2$:

Any $A \in S(F_2, G)$ has form $\begin{bmatrix} d_1 & 1 & 1 & 1 & 0 \\ 1 & d_2 & 1 & 1 & 0 \\ 1 & 1 & d_3 & 1 & 1 \\ 1 & 1 & 1 & d_4 & 1 \\ 0 & 0 & 1 & 1 & d_5 \end{bmatrix}$.

$A[145|235] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & d_5 \end{bmatrix}$ has determinant 1 so rank $A \geq 3$.

Therefore $\mathrm{mr}(F_2, G) \geq 3$.

Idea of Classification

To find the graphs characterizing $\{G \mid \mathrm{mr}(F, G) \leq k\}$:

1. Construct all matrices A of rank $\leq k$ over F.

$$A = U^t B U \iff \mathrm{rank}(A) \leq k.$$

   (B is $k \times k$, rank k; U is $k \times n$.)

2. Return non-isomorphic graphs of the matrices.

Problem: Too many matrices.

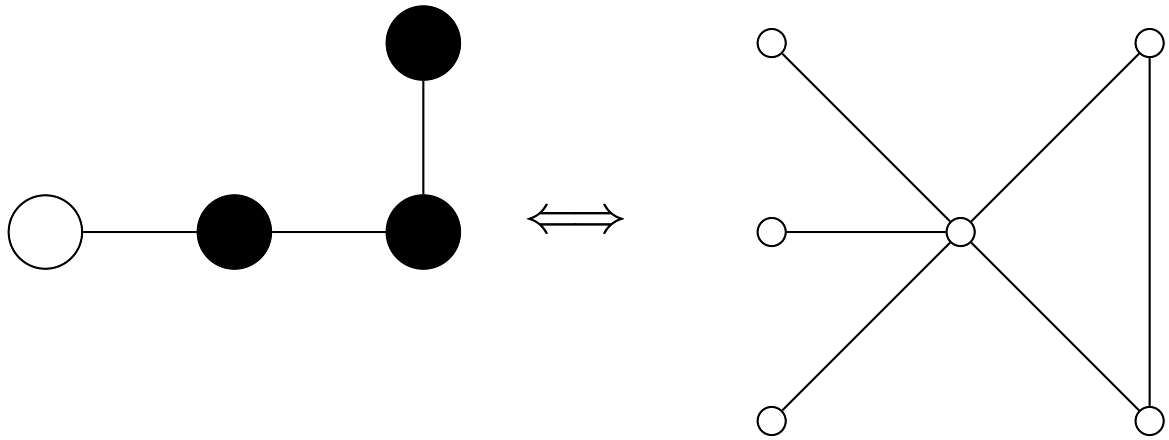
Solution: We only need the zero-nonzero patterns for the matrices. Be smarter by understanding $A = U^t B U$ better.

$$A = U^t B U,\ a_{ij} = (u_i, u_j)$$

Feature/operation on U	**Effect on graph corresponding to A**
Column in U	vertex in graph
Interchanging two columns	relabel vertices
Column in U isotropic wrt B	no loop at vertex (zero entry on diagonal)
Column in U not isotropic wrt B	loop at vertex (nonzero entry on diagonal)
Two columns orthogonal wrt B	no edge between corresponding vertices (zero matrix entry)
Two columns not orthogonal wrt B	edge between corresponding vertices (nonzero matrix entry)

$$A = U^t BU,\ a_{ij} = (u_i, u_j)$$

Feature/operation on U	**Effect on graph corresponding to A**
Duplicate isotropic columns	independent set, vertices have same neighbors
Duplicate non-isotropic columns	clique, vertices have same neighbors
Columns multiples of each other	corresponding vertices have same neighbors (remember, only the zero-nonzero pattern is needed, and there are no zero divisors in F)



Marked Graph $\qquad$ Substitution Graph

- white vertex (no loop) $\Longleftrightarrow$ isotropic $\Longleftrightarrow$ independent set

- black vertex (loop) $\Longleftrightarrow$ non-isotropic $\Longleftrightarrow$ clique

- edge $\Longleftrightarrow$ all possible edges

- cliques or independent sets can be empty

Algorithm

To find the graphs characterizing $\{G \mid \text{mr}(F, G) \le k\}$:

1. Columns of U are a maximal set of k-dimension vectors over F such that no vector is a multiple of any other.

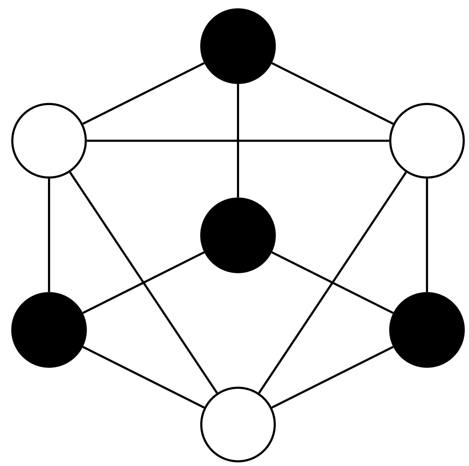
2. Construct all *interesting* matrices A of rank $\le k$.

$$A = U^t B U \iff \text{rank}(A) \le k.$$

   (B is $k \times k$, rank k; U is $k \times n$.)

3. Return non-isomorphic *marked* graphs of matrices.

Marked graph for $\mathrm{mr}(F_2, G) \leq 3$



Complement of incidence graph of Fano projective plane!

Projective Geometry

$V(k,q) = k$-dimensional vector space over F_q.

Equivalence relation on $V - \{\vec{0}\}$ by

$$x \sim y \iff x = cy, \quad \text{nonzero } c \in F.$$

Equivalence class $[x]$ is a line in V.

The points of projective geometry of dimension $k-1$ and order q, $PG(k-1,q)$, are equivalence classes $[x]$.

Projective Geometry

$[x] = \{cx \mid \text{nonzero } c \in F\}$

$q^k - 1$ vectors in $V(k,q) - \{\vec{0}\}$,

$q - 1$ vectors in each equivalence class,

so $\dfrac{q^k - 1}{q - 1}$ points in $PG(k-1, q)$.

Incidence Graph vs. Marked Graphs

Incidence Graph of Projective Geometry

Vertices: $[x] \in PG(k-1, q)$

Edges: $[x] — [y] \iff x^t y = 0$

Marked Graphs with $B = I_k$:

Vertices: $[x] \in PG(k-1, q)$

Edges: $[x] — [y] \iff x^t B y = x^t I_k y = x^t y \neq 0$

If $B = I_k$, marked graph is complement of incidence graph of $PG(k-1, q)$.

What about a different B?

Congruence Doesn't Change Marked Graph

$$A = U^t B U$$

C = change of basis matrix for $V(k,q)$.

$C^t BC$ **and** B **have same marked graph**

$U^t C^t BCU = (CU)^t B(CU)$ = basis transformation of U.

f : $[x] \mapsto [Cx]$. f is an isomorphism on equiv. classes.

f well defined: If $Cx = y$, then $C(kx) = kCx = ky \in [y]$.

f is surjective since C is nonsingular.

f is injective: $[Cx_1] = [Cx_2] \implies kCx_1 = Cx_2 \implies C(kx_1 - x_2) = 0 \implies kx_1 = x_2$ since C is nonsingular. This means $[x_1] = [x_2]$.

$C^t BC$ changes basis of columns of U, permuting columns of U, relabeling vertices of marked graph.

k-dim Bilinear Forms Over F_q, q odd

Up to congruence (change of basis), there are only two different k-dimensional bilinear forms B on F_q:

1. $B_1 = I_k$

2. $B_2 = I_{k-1} \oplus d$, d a nonsquare in F_q.

k odd $\implies C^t B_1 C = dB_2$.

Graph of B_2 = Graph of dB_2 = Graph of B_1

k odd: one marked graph, the complement of incidence graph of projective geometry $PG(k-1, q)$.

k even: two marked graphs, one is complement of incidence graph of projective geometry $PG(k-1, q)$.

Counting White Vertices in Marked Graphs

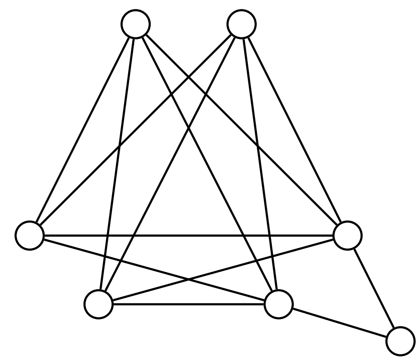
Using induction and representative bilinear forms, we get the following numbers of white vertices:

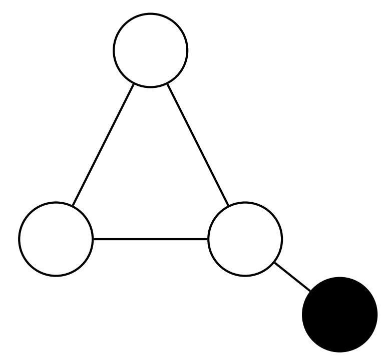
For odd $k = 2m + 1$: $\dfrac{q^{2m} - 1}{q - 1}$

For even $k = 2m$: $\dfrac{(q^m - 1)(q^{m-1} + 1)}{q - 1}$, $\quad \dfrac{(q^m + 1)(q^{m-1} - 1)}{q - 1}$

Marked Graphs for $\mathrm{mr}(F_3, G) \le k$

k	Vertices	White	Black
1	1	0	1
2	4	2	2
2	4	0	4
3	13	4	9
4	40	16	24
4	40	10	30
5	121	40	81
6	364	130	234
6	364	112	252
7	1093	364	729
8	3280	1120	2160
8	3280	1066	2214





Questions/Todo

1. Calculate marked graphs for even q.

2. Say more about the structure of the marked graphs. References?

3. What forbidden subgraphs characterize a given marked graph?